מאה שנים
של הכשרת מורים
מכללת לוינסקי לחינוך

בית הספר הגבוה לטכנולוגיה בירושלים
JERUSALEM COLLEGE OF TECHNOLOGY

הטכניון
מכון טכנולוגי לישראל
Technion
Israel Institute of Technology

A Theoretical- Computational Conflict

Thierry Dana-Picard

Jerusalem College of Technology

The first Jerusalem Conference
on RESEARCH in MATH EDUCATION
ט'-י' אדר תשע"ג
18-19 בפברואר 2013

מכון מופ"ת
בית ספר למחקר ולפיתוח תכניות
בהכשרת עובדי חינוך והוראה במכללות

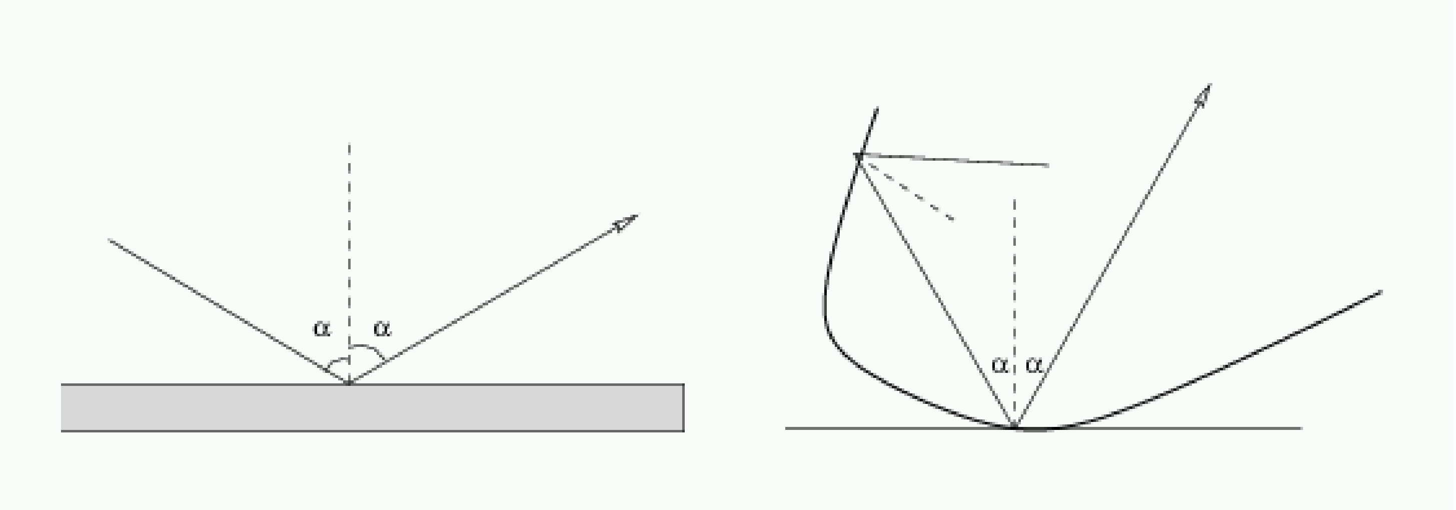
מטח
המרכז לטכנולוגיה חינוכית

The conflict

- Paths of light
- Polynomial equations
- Existence of solutions
- The CAS does not find the solutions

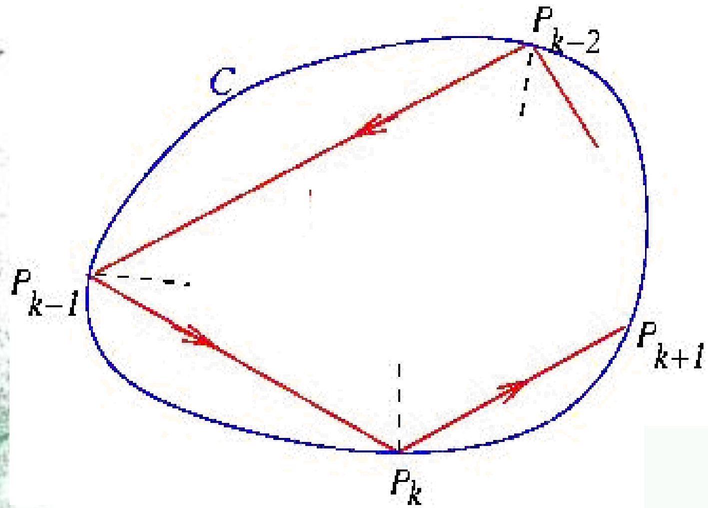
The conflict

The reflection law



The conflict

Closed paths of light

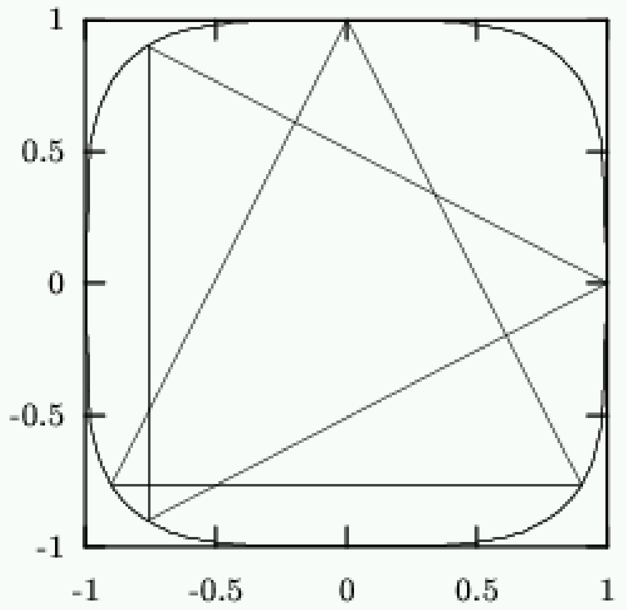


- Denote by $\overrightarrow{N_k}$ a vector normal to C at the point P_k .
- By the reflection law, we have:

$$\angle \overrightarrow{P_k P_{k-1}}, \overrightarrow{N_k} = \angle \overrightarrow{N_k}, \overrightarrow{P_k P_{k+1}}$$

The conflict

Triangles of light trapped in a closed Fermat curve



$$\cos(\overrightarrow{V},\overrightarrow{PB}) = \frac{-x_0 + (1-y_0)\cdot\left(\frac{y_0}{x_0}\right)^{n-1}}{\sqrt{1+\left(\frac{y_0}{x_0}\right)^{2n-2}}\cdot\sqrt{x_0^2+(1-y_0)^2}}$$

$$\cos(\overrightarrow{V},\overrightarrow{PQ}) = \frac{-2x_0}{\sqrt{1+\left(\frac{y_0}{x_0}\right)^{2n-2}}\cdot 2|x_0|}$$

The conflict

Polynomial equations

- General equations:

$$\begin{cases} x^n + y^n = 1 \\ (1-y)(y^{2n-2} - x^{2n-2}) - 2y^{n-1}x^n = 0 \end{cases}$$

- The case n=4

A Gröbner basis

$$\begin{aligned} g_1(x,y) &= x^4 + y^4 - 1 \\ g_2(x,y) &= 7x^2y - 7x^2 + 4y^{12} - 6y^{11} + 2y^{10} - 4y^9 \\ &\quad - 10y^8 + 23y^7 + y^6 + 6y^4 - 21y^3 + 7y^2 - 2 \\ g_3(x,y) &= 2y^{13} + 2y^{12} + 4y^{11} + 4y^{10} \\ &\quad - 3y^9 - y^8 - 4y^7 - 4y^6 + 3y^5 - 3y^4 - y + 1 \end{aligned}$$

The conflict

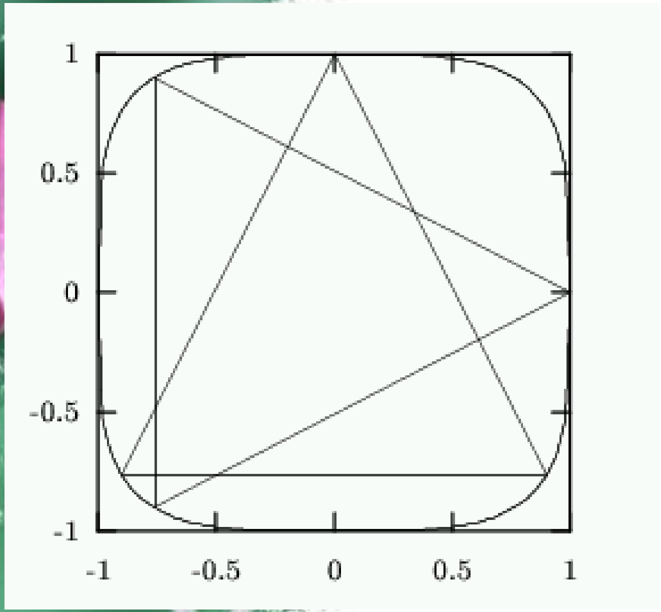
Where are the solutions?

- Intermediate Value Theorem implies that solutions exist
- The CAS **does not see** the solutions

The conflict

Why does the CAS not see the solutions?

- Fermat's Last Theorem ensures that the solutions must be irrational
- The algorithm works over the rational numbers



we are looking

for irrational numbers

using a rational device

Different roles for different ways of working

- Paper-and-pencil work: derive a mathematical proof of the *existence* of a solution.
- CAS-assisted part of the work: find an actual *construction* of a solution.

Consequence of the conflict between the theoretical result and the computer output

The student had to perform the following steps:

- Compare the results.
- Try to have a more profound insight into both processes, the paper-and-pencil one (theory) and the CAS assisted one.
- Solve the apparent contradiction.
- Dispatch the final answer in as complete a way as possible, either analytic or graphical.

Danger

A theoretical-computational conflict may lead the student to multiply technical tasks with the CAS (cf Trouche, 2004).

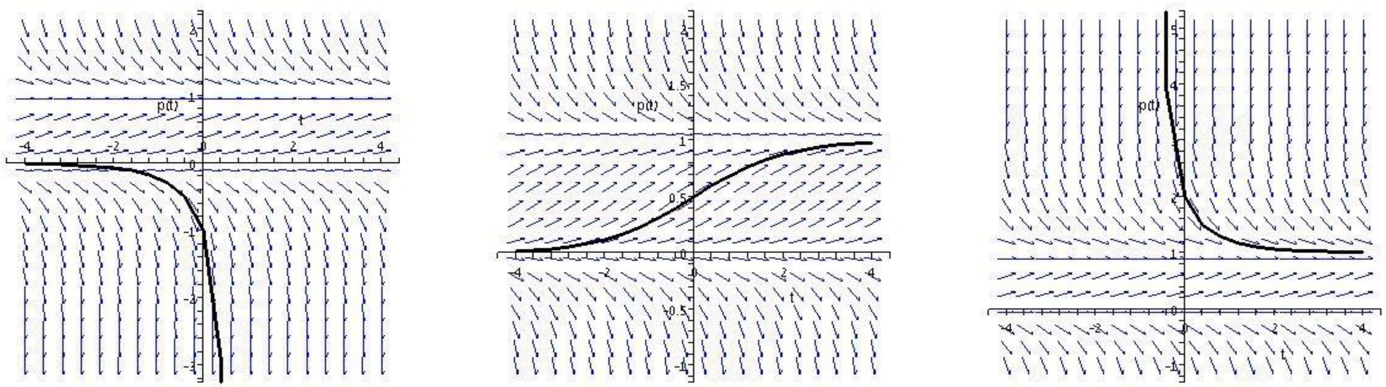
A conflict where one component is graphical

- A logistic differential equation (= a model for the evolution of a population)

$$\frac{dx}{dt} = rx\frac{K - x}{K}$$

where r is the intrinsic growth rate of the population and K is the "carrying capacity", i.e. the resources can carry a maximal population equal to K.

Example: $$y'(t) = y(t)\left(1 - y(t)\right)$$



What is specific
in our main example here?

It is a theoretical-computational conflict

where the computational part

is not graphical

but algebraic

What is the origin of the conflict?

- Tall-Vinner's terminology (1981):

  The student’s concept-image of an irrational number is not so clear at the beginning of his work.

- Zazkis's terminology (2005):

  Students have often only an opaque representation for irrationals.

(What about teachers? ☺)

Constraints

- *internal constraints* (linked to hardware)
- *command constraints* (linked to the existence and syntax of the commands)
- *organization constraints* (linked to the interface artifact-user)
- Here we have a motivating constraint

(cf Th. D-P. 2007)

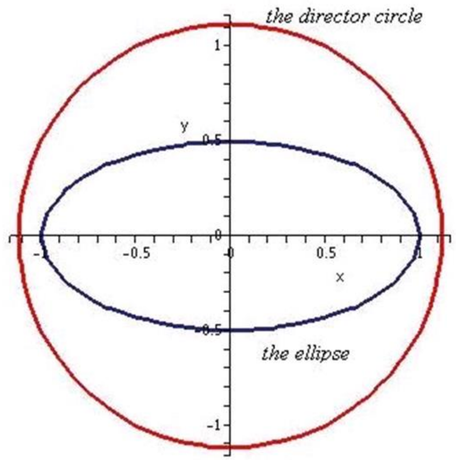
Motivating constraint

The conflict may lead to study more profoundly:

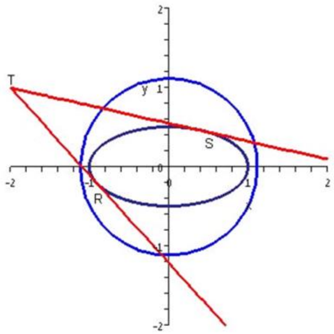
- *Rational vs irrational numbers*
- *Fermat Last Theorem*
- *Buchberger algorithm*
- *Etc.*

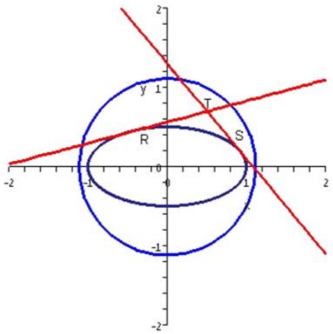
Another conflict: isoptic curves of an ellipse

- Orthoptic curve of an ellipse = director circle



- Viewing an ellipse under another angle

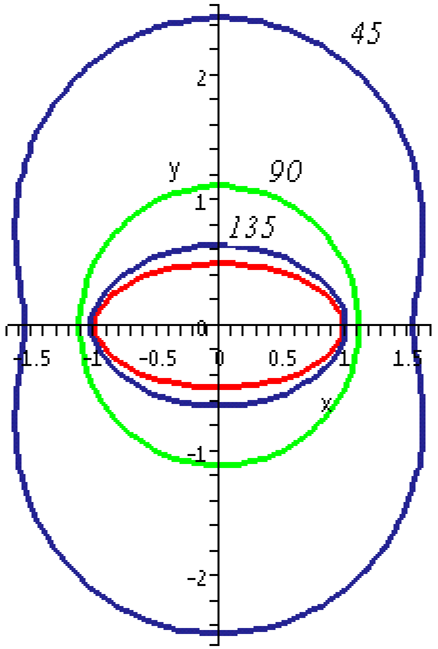




Bisoptics of an ellipse

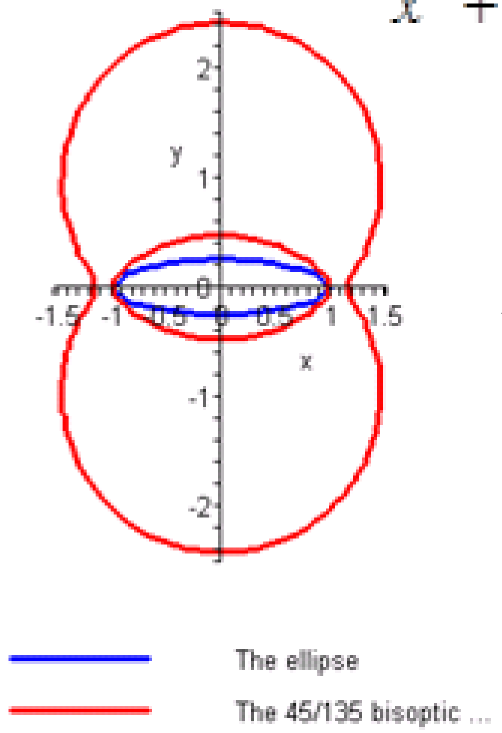
- The question: find the geometric locus of points from which a given ellipse is viewed under a given angle.
- The activity around the solution process:
  - *Algebraic computations, either by hand or using a CAS*
  - *Graphical representation of the obtained quartic equation which describes the solution*
  - *Look for an interpretation using an internet database*
  - *Solve the conflict!*

Examples of isoptic curves of an ellipse



An ellipse with two isoptic curves

$$x^4 + 2x^2y^2 + y^4 - \frac{19}{8}x^2 - \frac{49}{8}y^2 + \frac{353}{256} = 0$$



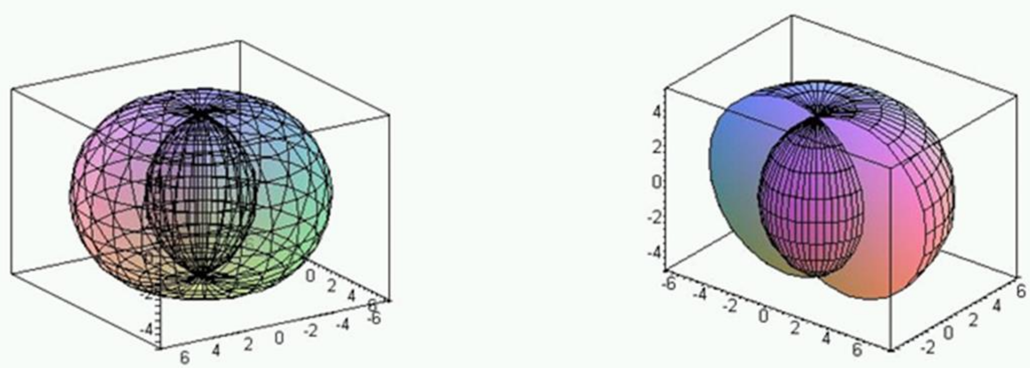
Conflict: the database does not fit the result of the computations and the plot

Wassenaar J. (2003a) Bicircular quartic, http://www.2dcurves.com/quartic/

Two kinds of tori

(a) Non self intersecting
R=5, *r*=2

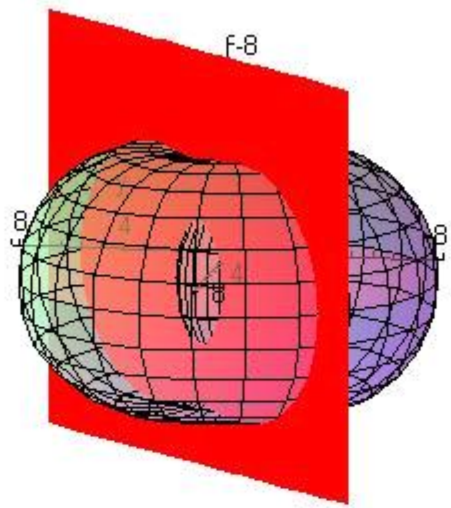
(b) self-intersecting
R=4, *r*=5

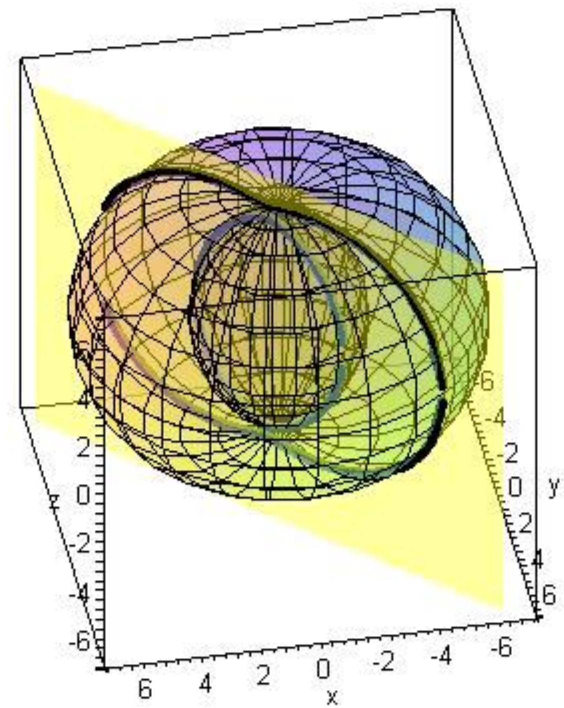


(a) The torus

(b)View from inside

Spiric curves=toric intersections





A more theoretical insight into the learning process

Th. Dana-Picard: *Existence theorems vs. Construction of a Solution: An example of a Theoretical-Computational Conflict*, ICME 11, July 2008.

N. Zehavi and G. Mann: *Development Process of a Praxeology for Supporting the Teaching of Proofs in a CAS Environment Based on Teachers' Experience in a Professional Development Course*, Technology, Knowledge and Learning, July 2011, Volume 16, Issue 2, pp 153-181.

Thanks to all the cognitive conflict makers all over the world