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# Response of a Loaded Idealized Piezoelectric Plate to an Electric Signal*

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The response of piezoelectric plates and rods are generally treated in the form of "equivalent" electric circuits. When the actual mechanical displacements and strains are of interest, such equivalent circuit treatment may be inconvenient. In the present paper the response of a loaded piezoelectric plate to an arbitrary electric input signal is derived on the basis of certain idealizations which are closely approximated by practical systems.

## A. INTRODUCTION

THE present exposition is an attempt to analyze the response of a piezoelectric plate to any applied field, when this plate is in contact with other acoustic media. A rigorous solution of this problem is, of course, too complex to attempt here. But it seems that by introducing a degree of idealization, the situation may be reduced to a very simple one-dimensional boundary value problem, whose solution will not differ greatly from the results to be expected in an actual system for a wide variety of practical conditions. As a result of this simplification the solution will, with minor modifications, be applicable also to other electromechanical transducers.

A survey of the literature shows considerable work on loaded piezoelectric crystals such as done primarily by Langevin and Biquard[1] and by Mason.[2] Compare also May.[3] But the former work was strictly on sinusoidal signals and all the later work seems to have been by the use of "equivalent electric circuits," which are highly appropriate when only the electrical characteristics are of interest, but are less readily applicable when the mechanical effects are being investigated. It is therefore hoped that the results presented here will serve a useful purpose.

The following simplifying assumptions will be made:

1. We are dealing with a transducer plate whose lateral dimensions far exceed its thickness, so that fields, strains, and stresses may all be considered plane.[4]
2. The shearing stress introduced by a compressional strain and the compressional stress introduced by a shearing strain will be negligible compared to the corresponding compressional and shearing stresses, respectively.[5]
3. The losses within the transducer are negligible compared to the energy radiated into the surrounding media.
4. The transducer is surrounded by homogeneous acoustic media, extending to infinity.
5. The displacements are maintained low enough, so that the transducer does not separate from the medium during any part of the cycle and that the elastic and electromechanical effects may be considered to be linear.

Although the foregoing conditions may seem rather ideal and will never be fully realized in a practical situation, the deviation will be negligible in many practical applications. In particular, the results will be applicable to many systems where high-frequency acoustic energy is used for information purposes, as in ultrasonic delay lines and ultrasonic light modulators. They are further justified by the fact that they make these situations readily amenable to an analysis which would otherwise probably be not practical.

Since the above conditions reduce the problem to a one-dimensional one, only the following parameters need be considered:

1. The compressional and shear strains ($2d$) introduced by a unit signal under static conditions with the transducer clamped laterally.
2. The restoring stress introduced by a unit compressional or shear strain, both in the transducer and in the media, i.e., the appropriate moduli of elasticity.
3. The densities ($\rho$) of the transducer material and the media.
4. The thickness ($2D$) of the transducer.

In the following, the ratio of the elastic modulus to the density is written as $v^2$, since it equals the square of the velocity with which a local disturbance would be propagated.

As a result of the above simplifying assumptions, a single differential equation results for each medium, namely $(\partial^2 a/\partial t^2)-v^2(\partial^2 a/\partial x^2)=0$, where $a$ is the displacement in the particular medium.

This set of differential equations will be solved for a unit-step input, since for this case the boundary conditions are most obvious physically. Differentiating the result with respect to time, yields the impulse response.

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1 P. Biquard, Rev. d'Acoustique 3, 104 (1934).
2 W. P. Mason, *Electromechanical Transducers and Wave Filters* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1942), 2nd ed., p. 230.
3 J. G. May, Jr., J. Acoust. Soc. Am. 26, 347 (1954).
4 In order for this assumption to apply rigorously the results must be limited to time intervals short with respect to the lateral dimension divided by the velocity of strain propagation in the transducer.
5 This is rigorously true, certainly for homogeneous magnetostrictive transducers, but also for the most common piezoelectric transducers, such as the x-cut quartz crystal where $c_{15}$ and $c_{16}$ vanish.

The convolution of this with any arbitrary input signal, will yield the corresponding response.

One set of boundary conditions is due to the fact that the displacement must be continuous across a boundary, as must be the stress (which equals $\rho v^2 \partial a/\partial x$). The system will be considered to be at rest with a negative unit potential applied across the crystal electrodes producing strain, $d$. The electrodes are then short circuited at time $t=0$. The resulting vibrations will then be strictly elastic ones, since the mechanical effects of the locally created electric polarization are included in the moduli of elasticity, $\rho v^2$. Thus the initial unit strain is assumed to be $-d/D$ in the transducer and vanishing elsewhere. The displacement at all finite times vanishes for infinite $|x|$ (or, in view of our boundary conditions at $t=0$, this displacement remains at $d$).

### B. DERIVATION OF IMPULSE AND STEP-FUNCTION RESPONSES

Our system consists of three media, limited in the $x$ direction and infinite in the $yz$ plane. Let the magnitude of vibration in these media be $a$, $b$, $c$, where $a$ refers to the piezoelectric transducer; $\rho$ and $v$, with the appropriate subscripts, refer respectively to the density of the medium and to the velocity of the disturbance for the three regions. Let $2D$ be the transducer thickness and $2d$ the initial total strain.

We have then the following equations and boundary conditions:

$-\infty<x<-D$:

$$(\partial^2 b/\partial x^2)-(1/v_b^2)(\partial^2 b/\partial t^2)=0; \tag{1}$$

$$b(0,x)=+d; \tag{2}$$

$$\dot{b}(0,x)=0; \tag{3}$$

$$b(t,-\infty)=d; \tag{4}$$

$-D<x<D$:

$$(\partial^2 a/\partial x^2)-(1/v_a^2)(\partial^2 a/\partial t^2)=0; \tag{5}$$

$$a(0,x)=-xd/D; \tag{6}$$

$$\dot{a}(0,x)=0; \tag{7}$$

$D<x<\infty$:

$$(\partial^2 c/\partial x^2)-(1/v_c)(\partial^2 c/\partial)t^2=0; \tag{8}$$

$$c(0,x)=-d; \tag{9}$$

$$\dot{c}(0,x)=0; \tag{10}$$

$$c(t,\infty)=-d; \tag{11}$$

$x=-D$:

$$a(t,-D)=b(t,-D); \tag{12}$$

$$\rho_a v_a^2[\partial a(t,-D)/\partial x]=\rho_b v_b^2[\partial b(t,-D)/\partial x]; \tag{13}$$

$x=D$:

$$a(t,D)=c(t,D); \tag{14}$$

$$\rho_a v_a^2[\partial a(t,D)/\partial x]=\rho_c v_c^2[\partial c(t,D)/\partial x]. \tag{15}$$

Note that the Laplace transforms are

$$\mathcal{L}(\partial^2 a/\partial t^2)=s^2 A-sa(0,x)-\dot{a}(0,x);$$
$$\mathcal{L}(\partial^2 a/\partial x^2)=\partial^2 A/\partial x^2.$$

Thus Eqs. (1), (5), and (8) become, respectively,

$$\partial^2 B/\partial x^2-(s^2 B-sd)/v_b^2=0, \tag{16}$$

$$\partial^2 A/\partial x^2-(s^2 A+sxd/D)/v_a^2=0, \tag{17}$$

$$\partial^2 C/\partial x^2-(s^2 C+sd)/v_c^2=0. \tag{18}$$

The solutions are, respectively, of the form

$$B=\beta_1 e^{sx/v_b}+\beta_2 e^{-sx/v_b}+d/s, \tag{19}$$

$$A=\alpha_1 e^{sx/v_a}+\alpha_2 e^{-sx/v_a}-xd/sD, \tag{20}$$

$$C=\gamma_1 e^{sx/v_c}+\gamma_2 e^{-sx/v_c}-d/s, \tag{21}$$

where the $\alpha$, $\beta$, $\gamma$ represent functions of $s$ determined by the boundary conditions. From conditions (4) and (11) it is apparent that $\beta_2=\gamma_1=0$. From Eqs. (12) and (14), by substituting Eqs. (19)–(21), we find

$$\alpha_1 e^{-sD/v_a}+\alpha_2 e^{sD/v_a}+(d/s)=\beta e^{-sD/v_b}+(d/s),$$

or

$$\beta=[\alpha_1 e^{-sD/v_a}+\alpha_2 e^{sD/v_a}]e^{sD/v_b}. \tag{22}$$

Similarly,

$$\alpha_1 e^{sD/v_a}+\alpha_2 e^{-sD/v_a}-(d/s)=\gamma e^{-sD/v_a}-(d/s),$$

or

$$\gamma=[\alpha_1 e^{sD/v_a}+\alpha_2 e^{-sD/v_a}]e^{sD/v_c}, \tag{23}$$

where we have set $\beta_1=\beta$ and $\gamma_2=\gamma$. Again, from Eqs. (13) and (15), substituting Eqs. (19)–(21), we find

$$\rho_a v_a^2[(s/v_a)\alpha_1 e^{-sD/v_a}-(s/v_a)\alpha_2 e^{sD/v_a}-(d/sD)]$$
$$=\rho_b v_b^2[(s/v_b)\beta e^{-sD/v_b}]$$

or, letting

$$\rho_a v_a/\rho_b v_b=m_1, \tag{24}$$

$$\beta=m_1[\alpha_1 e^{-sD/v_a}-\alpha_2 e^{sD/v_a}-(v_c d/s^2 D)]e^{sD/v_b}. \tag{25}$$

Similarly,

$$\rho_a v_a^2[(s/v_a)\alpha_1 e^{sD/v_a}-(s/v_a)\alpha_2 e^{-sD/v_a}-(d/sD)]$$
$$=\rho_c v_c^2[-(s/v_c)\gamma e^{-sD/v_c}]$$

or, letting

$$\rho_a v_a/\rho_c v_c=m_2, \tag{26}$$

$$\gamma=m_2[-\alpha_1 e^{sD/v_a}+\alpha_2 e^{-sD/v_a}+(v_a d/s^2 D)]e^{sD/v_c}. \tag{27}$$

Equations (22), (23), (25), and (27) involve four unknown functions of $s$: $\alpha_1$, $\alpha_2$, $\beta$, $\gamma$. Solving for these functions we obtain, after some straightforward but tedious manipulations, letting $\tau \equiv D/v_a$,

$$\alpha_1=\frac{2d}{s^2\tau}\left[\frac{m_1(1-m_2)+m_2(1+m_1)e^{s\tau}}{-(1-m_1)(1-m_2)+(1+m_1)(1+m_2)e^{2s\tau}}\right]e^{\frac{1}{2}s\tau}, \tag{28}$$

$$\alpha_2=\frac{-2d}{s^2\tau}\left[\frac{m_2(1-m_1)+m_1(1+m_2)e^{s\tau}}{-(1-m_1)(1-m_2)+(1+m_1)(1+m_2)e^{2s\tau}}\right]e^{\frac{1}{2}s\tau}, \tag{29}$$

$$\beta=\frac{2dm_1}{s^2\tau}\left[\frac{(1-m_2)+2m_2e^{s\tau}-(1+m_2)e^{2s\tau}}{-(1-m_1)(1-m_2)+(1+m_1)(1+m_2)e^{2s\tau}}\right]e^{sD/v_b}, \tag{30}$$

$$\gamma=\frac{2dm_2}{s^2\tau}\left[\frac{-(1-m_1)-2m_1e^{s\tau}+(1+m_1)e^{2s\tau}}{-(1-m_1)(1-m_2)+(1+m_1)(1+m_2)e^{2s\tau}}\right]e^{sD/v_c}. \tag{31}$$

After introducing the following simplifying definitions, involving only constants of the transducer and the media

$$K=2d/\tau(1+m_1)(1+m_2),$$
$$k=(1-m_1)(1-m_2)/(1+m_1)(1+m_2), \tag{32}$$

we may write

$$A=\frac{Ke^{s\tau/2}}{s^2(e^{2s\tau}-k)}\{[m_1(1-m_2)+m_2(1+m_1)e^{s\tau}]e^{sx/v_a}$$
$$+[m_2(1-m_1)+m_1(1+m_2)e^{s\tau}]e^{-sx/v_a}\}-\frac{xd}{sD}, \tag{33}$$

$$B=\frac{m_1Ke^{sD/v_b}}{s^2(e^{s\tau}-k)}[(1-m_2)+2m_2e^{s\tau}-(1+m_2)e^{2s\tau}]e^{sx/v_b}+\frac{d}{s}, \tag{34}$$

$$C=\frac{m_2Ke^{sD/v_c}}{s^2(e^{s\tau}-k)}[-(1-m_1)-2m_1e^{s\tau}+(1+m_1)e^{2s\tau}]$$
$$\times e^{-sx/v_c}-\frac{d}{s}. \tag{35}$$

These expressions represent the Laplace transform of the displacement of any part of the system at any time after a negative unit potential has been removed from the transducer. By subtracting the initial displacement, the response to a unit step is obtained. By multiplying these expressions by $s$, the Laplace transform of the system response to a unit impulse is obtained. The Laplace transform of the system response to any signal in time is obtained simply by multiplying the Laplace transform of the signal with $sA$, $sB$, or $sC$.

In considering the response of the medium we note that a change in the $x$ coordinate $\Delta x$ introduces only a time shift $\Delta t=\Delta x/v$. This is, of course, due to the fact that absorption losses in the medium have been neglected. For an analysis of the medium response it will therefore suffice to limit the discussion to Eq. (35) with $x=D$. Under these conditions, the impulse response of the system will be given by the inverse Laplace transform of

$$C_0'=sC_0=s^{-1}[m_2K/(e^{s\tau}-k)]$$
$$\times[(m_1+1)e^{2s\tau}-2m_1e^{s\tau}+(m_1-1)].$$

Since an expansion of

$$(e^{s\tau}-k)^{-1}=\sum_{n=1}^{\infty}k^{n-1}e^{-2ns\tau},$$

we may write

$$C_0'=\frac{m_2K}{s}[(m_1+1)e^{2s\tau}-2m_1e^{s\tau}+(m_1-1)]\sum_{n=1}^{\infty}k^{n-1}e^{-2ns\tau}.$$

By multiplying out and combining terms with equal exponents, we obtain

$$C_0'=\frac{m_2K}{s}\{[k(m_1+1)+(m_1-1)]\sum_{n=1}^{\infty}k^{n-1}e^{-2ns\tau}$$
$$-2m_1\sum_{n=1}^{\infty}k^{n-1}e^{-(2n-1)s\tau}+m_1+1\}. \tag{36}$$

The impulse response of the system at any time, that is the inverse Laplace transform of this expression, may readily be obtained by noting that $Ke^{-T}/s$ transforms into a step of height $K$ at time $T$.

Consequently, in order to obtain the magnitude of the response at any time $T$, it is necessary only to sum the coefficients of those exponentials with exponents less than $T$ in Eq. (36). Thus, for instance, for $2r\tau<t<(2r+1)\tau$ the displacement at the transducer face (on the positive side $m_2$) will be

$$c=m_2K(m_1+1)k^r.$$

For

$$(2r+1)\tau<t<2(r+1)\tau, \quad C=-m_2K(m_1-1)k^r. \tag{37}$$

The following special cases may be of interest:

1. The transducer is loaded on one side only. This condition corresponds to $\rho_b=0$ and therefore $m_1=\infty$.

Consequently,

$$K=2d/[\tau m_1(1+m_2)]\equiv K'/m_1,\quad k=[(m_2-1)/(m_2+1)]\equiv k_s. \tag{38}$$

The resulting expression for the Laplace transform of the impulse response is

$$C_0'=\frac{m_2K'}{s}[(k_s+1)\sum k_s^{n-1}e^{-2ns\tau}-2\sum_{n=1}^{\infty}k_s^{n-1}e^{(2n-1)s\tau}+1], \tag{39}$$

and for $2r\tau<t<(2r+1)\tau$,

$$c=m_2K'k_s^r.$$

For

$$(2r+1)\tau<t<2(r+1)\tau,\quad c=-m_2K'k_s^r. \tag{40}$$

2. The transducer is loaded equally on both sides. Then

$$K=2d/[\tau(1+m)^2]$$

and

$$k_d=[(1-m)/(1+m)]^2=k_s^2. \tag{41}$$

$$C_0'=\frac{mK}{s}\{[(m+1)k_s^2+m-1]\sum_{n=1}^{\infty}k_s^{2(n-1)}e^{-2ns\tau}-2m\sum_{n=1}^{\infty}k_s^{2(n-1)}e^{-(2n-1)s\tau}+m+1\}$$

$$=\frac{mK}{s}\{2m[k_s\sum k_s^{2(n-1)}e^{-2ns\tau}-\sum k_s^{2(n-1)}e^{-(2n-1)s\tau}]+m+1\}, \tag{42}$$

is the impulse response Laplace transform. Thus for $2r\tau<t<(2r+1)\tau$,

$$c=m(m+1)Kk_s^{2r}.$$

For

$$(2r+1)\tau<\tau<2(r+1)\tau,\quad c=-m(m-1)Kk_s^{2r}. \tag{43}$$

3. The transducer is completely unloaded. In this case the relations are the same as in the preceding case with $m$ approaching infinity. Thus

$$K=2d/\tau m^2,\quad k=1, \tag{44}$$

$$C_0'=\frac{2d}{\tau ms}[2m\sum e^{-2ns\tau}-2m\sum e^{-(2n-1)s\tau}+m]$$
$$=\frac{4d}{\tau s}[\sum e^{-2ns\tau}-\sum e^{-(2n-1)s\tau}+\tfrac{1}{2}]. \tag{45}$$

The impulse response is thus a square wave with double amplitude $4d/\tau$ and period $2\tau$. This result is due to the fact that the above results have been derived for a lossless transducer.

A graphic presentation of the displacement of the transducer face, as a function of time, is shown in Fig. 1. Figure 1(a) represents the response to a step-function and Fig. 1(b) the response to an impulse. For normalization purposes, the coefficient $m_1m_2K$ is assumed equal to unity and $m_1\gg 1$.

In the case of the step-function input we might have expected a net displacement $d$ at $t=\infty$. The fact that displacement vanishes at $t=\infty$ is due to the fact that our assumption of $m_1\gg 1$ corresponds to a transducer effectively unloaded on the rear surface so that the difference between initial and steady state strain will appear at the rear of the crystal.

In the case of $m_1=m_2$, the integral of the impulse response from $t=0$ to $t=\infty$ equals $d$, half the steady state total strain, as would be expected from symmetry considerations. This result can be obtained by summing Eqs. (43) for $r=0$ to $r=\infty$ and multiplying by $\tau$, corresponding to integration from $t=0$ to $t=\infty$.

In some instances the state of strain of the media may be of interest. For instance, the change of refractive index of the medium may be considered, to a first approximation, proportional to the strain.

The strain of the medium is equal to the rate of change of the displacement (with respect to the direction of propagation). Thus the Laplace transforms of the strains resulting from a step-function are, using Eqs. (33), (34) and (35):

$$A^*=\frac{\partial A}{\partial x}=\frac{Ke^{s\tau/2}}{sv_a(e^{2s\tau}-k)}\times\{[m_1(1-m_2)+m_2(1+m_1)e^{s\tau}]e^{sx/v_a}-[m_2(1-m_1)+m_1(1+m_2)e^{s\tau}]e^{-sx/v_a}\}-\frac{d}{sD}, \tag{46}$$

$$B^*=\frac{\partial B}{\partial x}=\frac{m_1Ke^{sD/v_b}}{sv_b(e^{2s\tau}-k)}\times[(1-m_2)+2m_2e^{s\tau}-(1+m_2)e^{2s\tau}]e^{sx/v_b}, \tag{47}$$

$$C^*=\frac{\partial C}{\partial x}=\frac{m_2Ke^{sD/v_c}}{sv_c(e^{s\tau}-k)}\times[(1-m_1)+2m_1e^{s\tau}-(1+m_1)e^{2s\tau}]e^{-sx/v_c}. \tag{48}$$

The corresponding expressions for an impulse input are obtained from the above simply by multiplying by $s$. Corresponding to Eq. (36), the expressions at the transducer face are for a step-function input and impulse input, respectively,

$$C_0^*=-\frac{m_2K}{v_cs}\{[k(m_1+1)+(m_1-1)]\sum_{n=1}^{\infty}k^{n-1}e^{-2ns\tau}-2m_1\sum_{n=1}^{\infty}k^{n-1}e^{-(2n-1)s\tau}+m_1+1\}, \tag{49}$$

$$C_0'^*=-\frac{m_2K}{v_c}\{[k(m_1+1)+(m_1-1)]\sum_{n=1}^{\infty}k^{n-1}e^{-2ns\tau}-2m_1\sum_{n=1}^{\infty}k^{n-1}e^{-(2n-1)s\tau}+m_1+1\}. \tag{50}$$

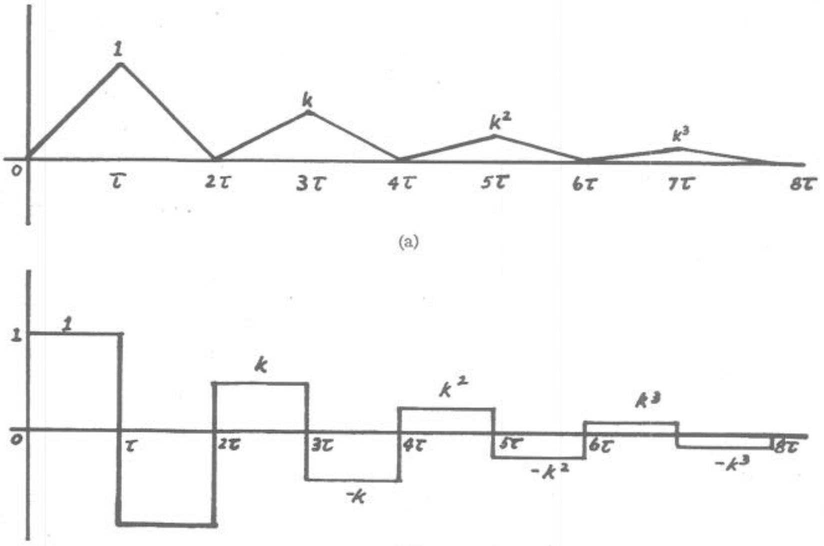


FIG. 1. Graphic presentation of the displacement of the transducer face, as a function of time. Part (a) represents the response to a step-function and part (b) is the response to an impulse.

The former is, within a factor $(-v_c)$, equal to the Laplace transform of the displacement due to an impulse and the latter represents a series of impulses of alternating polarity—the time derivative of the step-function represented by the former.

### C. RESPONSE TO A SINUSOIDAL INPUT

The steady-state response to a sinusoidal signal both in amplitude and phase angle is readily obtained from the impulse response, simply by substituting $(iw)$ for $(s)$ in the latter, where $w$ is the radian frequency of the applied signal.

In the case where $w=\pi/\tau$, this results in particularly simple expressions, which are listed below for reference. This condition corresponds, of course, to the case where the fundamental resonance frequency of the transducer is applied to its electrodes.

In the case of the non-symmetrically loaded transducer, the amplitude of the resulting displacement variations are for a signal $V=\sin\pi t/\tau$:

$$c_p=(4m_1m_2d)/\pi(m_1+m_2). \tag{51}$$

When one side only is loaded, the response is

$$c_p=4md/\pi. \tag{52}$$

When both sides are loaded equally

$$c_p=2md/\pi. \tag{53}$$

The corresponding expressions for the unit strain are

$$c_p^*=(4m_1m_2d)/v_c\tau(m_1+m_2), \tag{54}$$

$$c_p^*=4md/v_c\tau, \tag{55}$$

$$c_p^*=2md/v_c\tau. \tag{56}$$

### D. NUMERICAL EXAMPLE

If the transducer is an air-backed, x-cut quartz crystal, with resonant frequency 15 Mc, in contact with water, and if a square pulse (duration $T=0.11$ μsec, magnitude 1000 volts) is applied, we write the following equation for the strain at the transducer face as a function of time, referring to Eq. (49) and applying the simplifications incorporated in Eq. (39):

$$\mathcal{L}[S(t)]=\frac{2dm}{s\tau v_c(1+m)}\Big[(k_s+1)\sum_{n=1}^{\infty}k_s^{n-1}e^{-2ns\tau} -2\sum_{n=1}^{\infty}k_s^{n-1}e^{-(2n-1)s\tau}+1\Big](1-e^{sT}). \tag{57}$$

The factor $(1-e^{-Ts})$ takes into account that the applied signal may be viewed as resulting from the superposition of two step functions, equal magnitude but opposite sense, one being delayed with respect to the other by a time equal to the pulse duration $T$.

It can be seen that $S(t)$ represents the superposition of two "staircase" functions. One of these takes on the following values:

$$\begin{aligned} &-\frac{2dm}{\tau v_c(m+1)}k_s^r, \quad 2r<t<2r+1;\\ &\frac{2dm}{\tau v_c(m+1)}k_s^r, \quad 2r+1<t<2r+2. \end{aligned} \tag{58}$$

While the other one takes on the negative of these values $T$ seconds later. Thus, setting

$$2dm/[\tau v_c(m+1)]\equiv K_1, \tag{59}$$

the values of the first function during successive intervals $\tau$ are

$$-K_1,\ +K_1,\ -k_sK_1,\ k_sK_1,\ -k_s^2K_1,\ +k_s^2K_1,\ -k_s^3K_1,\ +k_s^3K_1,\ \cdots.$$

To determine the numerical values involved, the following constants of the quartz crystal are required:

modulus of elasticity: $c_{11}=86\times10^{10}$ d/cm$^2$,[6]

piezoelectric constant: $e_{11}=5.2\times10^4$ cm$^{-\frac{1}{2}}$ g$^{\frac{1}{2}}$ s$^{-1}$,[7]

density of the quartz: $\rho a=2.65$ g/cm$^3$.

The strain $d$, produced in an x-cut quartz plate, whose lateral dimensions exceed its thickness considerably, is different under static and dynamic conditions. Under static conditions it is

$$d_s=Vd_{11}, \tag{60}$$

whereas under dynamic conditions it is

$$d=V(e_{11}/c_{11})=V[d_{11}-d_{11}(c_{12}/c_{11})+d_{14}(c_{14}/c_{11})], \tag{61}$$

where $d_{11}$ is the piezoelectric modulus in the $x$ direction and $V$ is the applied voltage in electrostatic units (1 e.s.u. = 300 v). The difference is due to the fact that under static conditions the crystal will be strained laterally by the applied field whereas under dynamic conditions it must be considered clamped in those directions.[8] In our case, therefore,

$$d=(1000/300)(5.2\times10^4/86\times10^{10})=0.2\times10^{-6}\text{ cm}.$$

For water, $\rho_c=1$ g/cm$^3$, $v_c=1.5\times10^5$ cm/sec. For quartz, under the applicable conditions, $v=(c_{11}/\rho)^{\frac{1}{2}}$. Therefore

$$m=(\rho_a v_a/\rho_c v_c)=(2.65/1.5\times10^5)[86\times10^{10}/2.65]^{\frac{1}{2}}=10,$$

$$\tau=(2f_0)^{-1}=0.033\times10^{-6}\text{ sec}.$$

Therefore

$K_1=7.3\times10^{-5}$,

$k_s=0.82$,

$k_s^n=0.82$, 0.67, 0.55, 0.45, 0.37, 0.30, 0.25, 0.20, 0.17, 0.14, 0.11, 0.09, 0.08, 0.06, 0.05, 0.04, 0.03, 0.03, 0.02, 0.02,

for $n=1, 2, 3, \cdots, 20$.

The net result due to the total pulse is obtained by adding the values of the positive step-function to those of the negative step-function, delayed by 0.11 $\mu$sec.

If this pulse is repeated at 1.18 Mc, new step-functions must be added to every 0.85 $\mu$sec.

Comparing the amplitude of the initial pulse, $K_1$ Eq. (59), to the amplitude of strain variations due to a sinusoidal signal, Eq. (55), we find that the latter exceeds the former by a factor of $2(m+1)$ which equals 22 in our example.

The maximum pulse in our example occurs at the termination of the input pulse ($t=-0.11$ $\mu$sec) at which time the amplitude due to the start of the pulse is $(-0.82)$ so that the net value at that time is $-1.82$. This, then, corresponds to the amplitude due to a *sinusoidal* signal of

$$(1000/22)\times1.82/\sqrt{2}=58.5\text{ v rms}.$$

[6] W. P. Mason, *Piezoelectric Crystals* (Van Nostrand Company, New York, 1950), p. 84.

[7] W. G. Cady, *Piezoelectricity* (McGraw-Hill Book Company, New York, 1946).

[8] L. F. Epstein, W. A. M. Anderson, and L. R. Harden, J. Acoust. Soc. Am. 19, 248 (1947), Appendix.