Fitting a Bandlimited Signal to Given Points

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Abstract—One of the fundamental problems in communications is the transmission of a signal through a bandlimited channel. It should, therefore, be of great interest to find a general method of transmitting an arbitrarily close approximation to any *finite* signal through a channel which is arbitrarily bandlimited.

The present paper develops such a method and evaluates the cost in time and energy to accomplish this feat.

The problem of fitting an arbitrarily bandlimited signal to a finite number of arbitrary points is solved, and the minimum energy signal, fitting the points and having a spectrum confined to the given pass band, is found. The behavior of this signal, with shrinking bandwidth, is investigated.

Introduction

One of the fundamental problems in communications is the transmission of a signal through a bandlimited channel. It should, therefore, be of great interest to find a general method of transmitting an arbitrarily close approximation to any *finite* signal through a channel which is arbitrarily bandlimited.

We show here how this can be done, by solving the following general problem: Given a finite number of points, disposed over a finite distance (or time) and a frequency band, find a function confined to the frequency band and passing through the given points. Of all the possible solutions, we seek the one with the minimum "energy" (integral of the function squared).

In connection with fitting a bandlimited signal to given sample points, the sampling theorem states that the signal is uniquely determined when the sample point spacing is

$$\Delta t = \frac{\pi}{b}, \frac{2\pi}{b} \tag{1}$$

for low-pass and band-pass spectra, respectively; that is to say that, given any set of points $(y_k, k\Delta t)$, $k = \cdots -2, -1, 0, 1, 2 \cdots$, one and only one signal $f(t)$ will pass through these points and have its spectrum limited to a width b. If Δt is made smaller than the value given in (1), there will, in general, be no such signal passing through the given points.

This theorem applies, however, only when the samples extend over all time. When they extend over only a finite interval, an arbitrarily bandlimited signal can be made to pass through any given set of points.

In the following, such a signal is determined for the general case. Specifically, of all the signals passing through the given points and satisfying the bandwidth restriction, that signal is found which has the minimum energy content.

Expressions are then developed for two special cases involving equispaced samples.

1) Sampling rate equal to that called for by the sampling theorem (the "Nyquist rate").
2) Low-pass signal with sample spacing much closer than the reciprocal of the bandwidth.

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General Case

Let $f(t)$ be a real function to be fitted to the points (y_i, t_i) $j = 1, 2, 3 \cdots n$. The radian frequency spectrum of this function is to be bounded $\omega_0 - b < |\omega| < \omega_0 + b$, $b \leq \omega_0$. Its Fourier transform may be written

$$\begin{aligned} \varphi(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt \\ &= \varphi_c(\omega) + i\varphi_s(\omega), \quad \omega_0 - b < |\omega| < \omega_0 + b \\ &= 0 \qquad |\omega| < \omega_0 - b,\ |\omega| > \omega_0 + b, \end{aligned} \tag{2}$$

where we have set

$$\varphi_c(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t\, dt \tag{3}$$

$$\varphi_s(\omega) = \int_{-\infty}^{\infty} f(t) \sin \omega t\, dt. \tag{4}$$

The inverse transform of $\varphi(\omega)$ yields the function $f(t)$. Since this is real, $\varphi_c(\omega)$ must be an even function and $\varphi_s(\omega)$ an odd function of ω.

We wish to minimize the signal "energy"

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\varphi(\omega)|^2\, d\omega \\ &= \frac{1}{\pi} \int_{\omega_0 - b}^{\omega_0 + b} [\varphi_c^2(\omega) + \varphi_s^2(\omega)]\, d\omega, \end{aligned} \tag{5}$$

while, simultaneously, fulfilling the n conditions

$$y_i = \frac{1}{\pi} \int_{\omega_0 - b}^{\omega_0 + b} \varphi(\omega) e^{-i\omega t_i}\, d\omega, \quad j = 1, 2, 3 \cdots n.$$

Separating the real and imaginary parts of φ and the exponential we note that the integrals of the imaginary terms vanish, and we may write

$$y_i = \frac{1}{\pi} \int_{\omega_0 - b}^{\omega_0 + b} [\varphi_c(\omega) \cos \omega t_i + \varphi_s(\omega) \sin \omega t_i]\, d\omega. \tag{6}$$

Using Lagrangian multipliers, the problem may now be written as the minimization of

$$E' = 2 \int_{\omega_0-b}^{\omega_0+b} \left\{ \varphi_c^2(\omega) + \varphi_s^2(\omega) - \pi \sum_{j=1}^{n} \lambda_j [\varphi_c(\omega) \cos \omega t_j + \varphi_s(\omega) \sin \omega t_j] \right\} d\omega.$$

Since the derivatives φ_c', φ_s' do not occur, Euler's equations here are equivalent to the vanishing separately of the derivatives with respect to φ_c and φ_s. This leads to

$$\varphi_c = \frac{\pi}{2} \sum_j \lambda_j \cos \omega t_j$$

$$\varphi_s = \frac{\pi}{2} \sum_j \lambda_j \sin \omega t_j. \tag{7}$$

To find the λ_j, we substitute (7) into (6) finding

$$y_j = \sum_i \sin b(t_j - t_i) \cos [\omega_0(t_j - t_i)]/(t_i - t_j) \tag{8}$$

where we define

$$\left. \frac{\sin b(t_j - t_i)}{t_i - t_j} \right|_{i=j} = b,$$

its limit as $t_j \to t_i$. The λ_i are then given by solving (8)

$$\lambda_i = \sum_j (-)^{i+j} y_j \frac{D_{ji}}{|D|}, \tag{9}$$

when

$$|D| \neq 0.$$

Here we have written

$$|D| = \left| \frac{\sin b(t_j - t_i)}{t_j - t_i} \cos \omega_0(t_i - t_i) \right|, \tag{10}$$

the expression between the bars being the typical element of the determinant, and D_{ij} the minor of

$$\frac{\sin b(t_j - t_i)}{t_j - t_i} \cos \omega_0(t_j - t_i)$$

in this determinant.

The existence of $|D|$ is assumed from here on. Substituting (7) into (5)

$$E = \frac{\pi}{2} \sum_i \sum_j \lambda_i \lambda_j \frac{\sin b(t_i - t_j)}{t_i - t_j} \cos \omega_0(t_i - t_j). \tag{11}$$

The function $f(t)$ is obtained from

$$f(t) = \frac{1}{2\pi} \int_{-(\omega_0+b)}^{\omega_0+b} - \int_{-(\omega_0-b)}^{\omega_0-b} [\varphi_c(\omega) + i\varphi_s(\omega)] \cdot [\cos \omega t - i \sin \omega t] \, d\omega,$$

where the values of φ_c, φ_s are given in (7). Symmetry considerations show that the imaginary terms vanish, so that

$$f(t) = \sum_i \lambda_i \frac{\sin b(t - t_i)}{t - t_i} \cos \omega_0(t - t_i). \tag{12}$$

In the case of $n = 1$, (9) is not applicable. Equation (8) becomes

$$y_1 = \lambda b \tag{8'}$$

and hence

$$\lambda = y_1/b \tag{9'}$$

$$E = \frac{\pi}{2} \lambda^2 b = \pi y_1^2/2b \tag{11'}$$

$$f(t) = y_1 \frac{\sin bt}{bt} \cos \omega_0 t, \tag{12'}$$

where we have set $t_1 = 0$ without loss of generality.

To apply the results to the low-pass case, we merely set $\omega_0 = 0$. This yields

$$\lambda_{i0} = 2 \sum_j (-)^{i+j} y_j D_{ji}/|D| \tag{9''}$$

$$E_0 = \frac{\pi}{4} \sum_i \sum_j \lambda_i \lambda_j \frac{\sin b(t_i - t_j)}{t_i - t_j} \tag{11''}$$

$$f_0(t) = \tfrac{1}{2} \sum_i \lambda_i \frac{\sin b(t - t_i)}{t - t_i}. \tag{12''}$$

Nyquist Rate Sampling

1) Consider the sampling points chosen so as to satisfy

$$t_j - t_i = m_{ij}\pi/b \tag{13}$$

where m_{ij} is an integer, for all $i, j = 1, 2, 3 \cdots n$.

When this condition is met, all the off-diagonal elements in the determinant $|D|$ vanish and

$$|D| = b^n \tag{14}$$

$$\lambda_i = y_i/b \tag{15}$$

$$E = \frac{\pi}{2b} \sum_i y_i^2 = \frac{\pi n}{2b} \overline{y^2} \tag{16}$$

$$f(t) = \frac{1}{b} \sum_i y_i \frac{\sin (t_i - t)b}{t_i - t} \cos \omega_0(t_i - t). \tag{17}$$

2) Equispaced samples satisfying condition (13) lead to

$$b\Delta t(i - j)/\pi = m_{ij}. \tag{18}$$

To satisfy this for $(i - j = 1)$,

$$\frac{b\Delta t}{\pi} = m. \tag{19}$$

With $m = 1$, this corresponds to Nyquist rate sampling.

Narrow-Band Low-Pass Signal

In the narrow-band case, the determinant $|D|$ (10) becomes

$$\lim_{b \to 0} |D| = |b \cos \omega_0(t_j - t_i)| = b^n |\cos \omega_0(t_j - t_i)|,$$

the expression between the bars again being the typical element. This determinant can readily be shown to vanish for any spacing for $n = 3$ and for equispaced samples for any $n > 2$. This implies that both the amplitude of the minimum energy function and its energy become very large as the bandwidth becomes very small, unless the numerator of λ_i vanishes. That is, unless

$$\sum_{j} (-)^{i+j} y_j D_{ji} = 0.$$

The behavior of the function and its energy with shrinking bandwidth was analyzed far more quantitatively for the case where the sampling spacing (Δt) is an integral multiple of the period corresponding to the center of the pass band, including the low-pass signal. Under these conditions

$$\omega_0 \Delta t = 2m\pi, \quad \cos \omega_0(t_i - t_j) = \cos 2m\pi(i - j) = 1,$$

and the typical element of the determinant $|D|$ becomes

$$\frac{\sin \beta(i - j)}{\Delta t(i - j)}$$

where

$$\beta = b\Delta t.$$

As the bandwidth becomes very small, it is convenient to expand the elements of the determinant in powers of β and to write

$$|D| = \frac{\beta^n}{\Delta t^n} |B|$$

where the typical element of B is

$$b_{ij} = \sum_{k=0}^{\infty} \frac{(-)^k}{(2k + 1)!} (i - j)^{2k} \beta^{2k}.$$

When the zero-order approximation is used, the b_{ij} all becomes unity, and $|B|$ vanishes for $n > 1$. In general, we show in Appendix I that the order of approximation used must exceed $2n - 3$. Indeed, we show there that, in the limit, $|D|$ can be written

$$\Delta t^n |D| \approx b(n)\beta^{-n^2}, \; \beta \ll 1$$

where $b(n)$ is a numerical constant, a function only of n.

The primary minors of $|D|$ are there shown to be

$$\Delta t^{n-1} D_{ii} \approx b_{ii}(n)\beta^{-(1/2)(n^2-n)+\epsilon}$$

where b_{ii} is again a numerical constant and

$$\epsilon = 0; n = 1, 2 \bmod 4,$$

$$\epsilon = 1; n = 0, 3 \bmod 4.$$

Hence, the factor D_{ii}/D, occurring in the expressions for the function and its energy, can be written

$$\frac{D_{ii}}{|D|} \approx \Delta t \frac{b_{ii}(n)}{b(n)} \beta^{-\alpha}, \; \alpha = \tfrac{1}{2}(n^2 + n - 2\epsilon), \; \beta \ll 1.$$

Table I shows the values of α for the first ten values of n.

For the low-pass case the minimum energy for equispaced samples was evaluated for $n = 1, 2$ in the limit as $\beta \to 0$.

TABLE I

n	2	3	4	5	6	7	8	9	10	11
α	3	5	9	15	21	27	35	45	55	65

The following results were obtained:

$$n = 1, E_0 = \pi y^2/b = \pi y^2 \Delta t/\beta$$

$$n = 2, E_0 = 3\pi(y_1 - y_2)^2 \Delta t/\beta^3, \; y_1 \neq y_2.$$

$f(t)$ has been plotted for the cases $n = 3$, $y_j = (-1)^j$, $t_j = j - 2$: $\beta = 0.1, \pi, 10\pi$.

See Figs. 1 and 2.

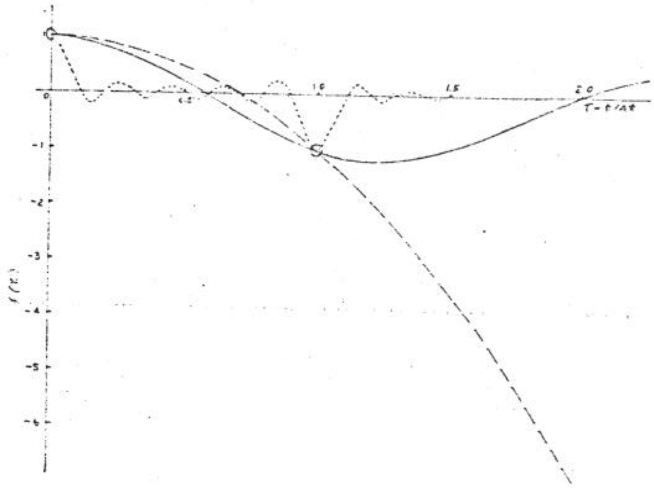

Fig. 1. Minimum energy signal fitted to points: $(-1, -1)$; $(0, 1)$; $(1, -1)$. Cutoff frequencies:
- - - - - - - - - $5/\Delta t$, $\beta = 10\pi$
———— $1/2\Delta t$, $\beta = \pi$
— — — — $1/20\pi\Delta t$, $\beta = 0.1$.

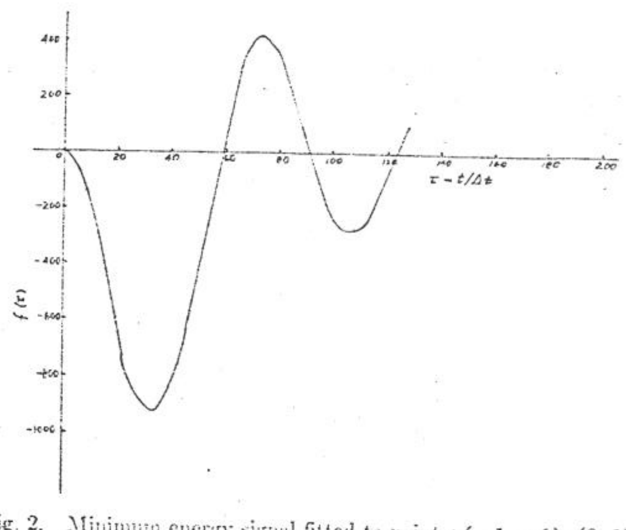

Fig. 2. Minimum energy signal fitted to points: $(-1, -1)$; $(0, 1)$; $(1, -1)$. Cutoff frequency: $1/20\pi\Delta t$.

Appendix I

Analysis of a Certain Determinant

Introduction

Here we shall investigate the behavior of a certain class of determinants. Specifically, we treat those determinants $|A|$ whose typical element is

$$a_{ij} = \sum_{k=0}^{\infty} c_k(i-j)^{2k}\beta^{2k}, \quad i, j = 1, 2, 3 \cdots n. \quad (20)$$

We seek the form this determinant takes as β approaches zero.

Since the typical element of **B** is

$$b_{ij} = \frac{\sin(i-j)\beta}{(i-j)\beta}, \quad (21)$$

$$\mathbf{B} = \mathbf{A}$$

with

$$c_k = \frac{(-)^k}{(2k+1)!}. \quad (22)$$

Value of the Determinant

Since we are interested only in the limit, we seek in the power expansion (in β) of the determinant only the term of lowest power with nonvanishing coefficient.

However, if we would use only the zero-order term in the expansion of (20), every element in it would equal c_0, and the determinant would vanish for $n > 1$.

We now describe a procedure by means of which we can eliminate a large number of the lower power terms in the individual elements without changing the value of the determinant. If this procedure leaves a nonvanishing determinant, we should have determined the behavior of $|\mathbf{A}|$ in the limit and should also have arrived at a good method for evaluating it for small β.

We note that the value of a determinant remains unchanged when one of its columns is subtracted from another, element by element.

For instance, by subtracting each column from its successor in this manner, the value of the determinant remains unchanged. We now repeat the process, omitting the first column, which had not been affected by the first step. We then repeat this process $(n-3)$ more times, omitting the first $(i-1)$ columns the ith time the process is applied. From this it follows that the elements in an nth-order determinant may be replaced in this manner: The jth element in each row is replaced by the $(j-1)$th difference of the series formed by the elements in this row.

As an example, consider a determinant in which the elements in each row form a first-order arithmetic progression. After the previously mentioned procedure is applied to it, the elements in the first column will be unchanged, the elements in the second column will all have their values equal to the difference of the progression, and all other elements will vanish. Thus, the value of this determinant will vanish for $n > 2$.

We now note a fundamental result of the calculus of finite differences, namely, that the jth differences of the series $1^j, 2^j, 3^j \cdots$, are all equal to $j!$ and the $(j+1)$th differences consequently vanish.

Now, the coefficients of β^{2k} in each row and each column form such a $(2k)$th-order arithmetic progression. Applying the previously mentioned differencing method, first to each row and then to each column, we find that each element may be replaced by

$$a'_{ij} = \sum_k c_k c_{ij,k} \beta^{2k} \quad (23)$$

where $c_{ij,k}$ is $(i+j-2)$th difference of the $(2k)$th-order progression $(i-j)^{2k}$. The power of the lowest order nonvanishing term is, then, $(i+j-2)$ and $(i+j-1)$ for even and odd $(i+j)$, respectively.

Since we are investigating $|\mathbf{A}|$ for $\beta \to 0$, only the lowest order nonvanishing term of the power series need be considered in each element (unless the resulting term vanishes). That is,

$$a'_{ij} \approx c_k c_{ij,k} \beta^{2k}, \quad 2k = i+j-2,\ i+j \text{ even}$$
$$i+j-1,\ i+j \text{ odd}. \quad (24)$$

We note for future reference that for $(i+j)$ even, $c_{ij,k} = (2k)!$

On expanding the determinant each term will contain each value of i and each value of j once and only once. The exponent of β in each term may, therefore, be written

$$\gamma = \sum_{i=1}^{n} i + \sum_{j=1}^{n} j - 2n + r \quad (25)$$

where r is the number of factors in the term for which $(i+j)$ is odd.

Since only the lowest order of β is of interest, we consider only terms for which $r = 0$ and hence,

$$\gamma_{mn} = n(n-1) \quad (26)$$

and the value of the determinant may be written

$$|\mathbf{A}| \approx |a'_{ij}|\, \beta^{-\gamma} \quad (27)$$

where the typical element is

$$a'_{ij} = (i+j-2)!c_{i+j-2}, \quad i+j \text{ even}$$
$$= 0, \quad i+j \text{ odd} \quad (28)$$

and

$$|\mathbf{D}| = \frac{\beta^n}{\Delta t^n}|\mathbf{B}| = \frac{|b'_{ij}|}{\Delta t^n}\beta^{-n^2},$$

$$b'_{ij} = (2i+2j-3)^{-1};\ i+j \text{ even}$$
$$= 0 \quad i+j \text{ odd}. \quad (28)$$

Although no proof for the existence of b'_{ij} in general has been found, it has been found to exist up to $n = 7$.

Primary Minors

We now seek the lowest value of the exponents appearing in the power expansion of the determinant A_{pq}, the general primary minor of $|\mathbf{A}|$.

The deletion of the pth row and qth column leaves the following four matrices of orders:

1) $(p - 1), (q - 1)$
2) $(p - 1), (n - q)$
3) $(n - p), (n - q)$
4) $(n - p), (q - 1)$, respectively,

when taken in clockwise order. Each of these consists of elements in the form of power series, with the coefficients of the k-power-term forming a kth-order arithmetic progression in each row and each column.

After applying the "differencing" procedure described earlier to each of these matrices, the lowest order term in each element a'_{ij} will have the following exponents:

1) $0 < i < p, 0 < j < q$:

$$a'_{ij} = i + j - 2, \ (i + j) \text{ even}$$
$$= i + j - 1, \ (i + j) \text{ odd}$$

2) $0 < i < p, q \leq j < n$:

$$a'_{ij} = i + j - q, \qquad (i + j + q) \text{ even}$$
$$= i + j - q - 1, \ (i + j + q) \text{ odd}$$

3) $d \leq i < n, q \leq j < n$:

$$a'_{ij} = i + j - p - q, \qquad (i + j + p + q) \text{ even}$$
$$= i + j - p - q + 1. \ (i + j + p + q) \text{ odd}$$

4) $p \leq i < n, 0 < j < q$:

$$a'_{ij} = i + j - p, \qquad (i + j + p) \text{ even}$$
$$= i + j - p - 1, \ (i + j + p) \text{ odd}.$$

We also note that the coefficients of the controlling terms in the $(q + s)$th column are identical to the corresponding ones in the $(s + 1)$th column, for $s = 0, 1, 2, 3, \cdots, (n - q - 1)$[1] for those elements for which $(i + j)$, $(i + j + p + 1)$ are even for $i < p$, $i \geq p$, respectively.

On subtracting the $(s + 1)$th columns from the $(q + s)$th columns, the controlling terms in the elements in the second quadrant with $(i + j + q)$ odd and those in the third quadrant with $(i + j + p + q)$ even will have their exponents increased by two.

The minimum values of a'_{ij} may thus be written in the following array, (Table II), where the value has been placed in the applicable quadrant and the upper sums correspond to even sums of i, j, p, q and the lower ones to odd sums of these. (In each quadrant the lesser value is underlined.)

It is readily shown that if, in forming one of the terms of the determinant, r elements are chosen from the first quadrant, contributions from the various quadrants will be as tabulated in Table III.

[1] We treat here the case of $q > n/2$; when $q \leq n/2$, the same method applies after the sequence of the columns has been inverted, a transformation which will not affect the absolute value of the determinant.

TABLE II

$i + j - 2$ $i + j - 1$	1 2	$i + j - q$ $i + j - q + 1$
$i + j - p$ $i + j - p - 1$	4 3	$i + j - p - q + 2$ $i + j - p - q + 1$

TABLE III

Quadrant Number	Number of Factors Contributed
1	r
2	$p - r - 1$
3	$n + r - p - q + 1$
4	$q - r - 1$

By multiplying each of the underlined values in Table II with the corresponding numbers of factors contributed, as shown in Table III, the lower limit of the exponent is found to be

$$\gamma'_{mn} \geq \sum_{i=1}^{n-1} i + \sum_{j=1}^{n-1} j$$
$$- 2r - q(p - r - 1) - (p + q - 1)$$
$$\cdot (n + r - p - q + 1) - (p + 1)(q - r - 1)$$

or

$$\gamma'_{mn} = n^2 + p^2 + q^2 - (n + 2)(p + q) - p + 2\{+1\},$$

where the term in braces is added when required to make γ'_{mn} even. (This means that one element in one quadrant must be chosen from the un-underlined set.)

Setting the p-derivatives and q-derivatives of this expression equal to zero shows that the lower limit of the exponent attains its minimum when p and q equal $(n + 1)/2$, $(n + 2)/2$, respectively. Since p and q are integers, these two conditions can never be satisfied simultaneously. It is found that the minimum lower limit is obtained when $p = q = (n + 2)/2$, $(n + 1)/2$ for even and odd n, respectively. The value obtained then is

$$\gamma'_{mn} = \tfrac{1}{2}(n - 1)(n - 2)\{+1\}$$

where the term in braces is added when required to make γ'_{mn} even.

We conclude that, in the limit the β-exponent of the corresponding minor of the determinant $|\mathbf{D}|$ is

$$\delta'_{mn} = \gamma'_{mn} + n - 1 = \tfrac{1}{2}n(n - 1) + \epsilon$$

where

$$\epsilon = 1, n = 0, 3 \bmod 4$$
$$= 0, n = 1, 2 \bmod 4.$$

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