OPTICA ACTA, 1970, VOL. 17, NO. 11, 869–872

On combined spatial and temporal characteristics of optical systems

LEO LEVI

Department of Physics, The City College,
City University of New York†

(*Received* 5 *June* 1970)

Abstract. Some optical imaging systems exhibit a response which varies with both spatial and temporal frequencies. Here, methods for treating such systems are presented and applied to simple examples.

The concepts of point-impulse response and spatio-temporal optical transfer function are introduced.

1. Fundamental considerations

In devices such as image tubes, both spatial and temporal characteristics are involved simultaneously. Assuming a linear response for the system, these may be treated by means of a three-dimensional Fourier transform. We limit our discussions here to the usual ' energy ' spread and transfer functions, which are applicable to radiation which is essentially incoherent—both spatially and temporally, i.e. when the time elements are large relative to the electromagnetic period and the space elements large relative to the coherence region. In the following, the terms ' vanishingly small ' and ' vanishingly short ' should be taken in that sense. The terms 'spread' and 'transfer' functions are occasionally applied also to the amplitude characteristics, a treatment appropriate in regions of coherence. That problem has been treated previously in a three-dimensional format [1].

In our incoherent case, the response of the system to a vanishingly short and vanishingly small input signal, may be termed the *point-impulse* response, described by

$$P_3(x, y, t).$$

Its three-dimensional Fourier transform would be the corresponding transfer function

$$T_3(\nu_x, \nu_y, \nu_t) = \int\int\int_{-\infty}^{\infty} P_3(x, y, t) \exp [i2\pi(\nu_x x + \nu_y y + \nu_t t)]\, dx\, dy\, dt, \tag{1}$$

where ν_x, ν_y are the Cartesian spatial frequency components, and ν_t is the temporal frequency. This may be called the spatio-temporal optical transfer function.

† Present address : Jerusalem School of Applied Sciences, 12 Hillel Street, Jerusalem, Israel.

Similarly, signals (and noise) can be described in the form $s(x, y, t)$. The corresponding spectrum, again, is given by the three-dimensional Fourier transform, as in (1). The auto-correlation function is given by:

$$C(x, y, t) \equiv s \odot s \equiv \lim_{X, Y, T \to \infty} \frac{1}{XYT} \int_0^T \int_0^Y \int_0^X \times s(x', y', t') s(x'-x, y'-y, t'-t)\, dx'\, dy'\, dt'. \quad (2)$$

The corresponding Wiener spectrum is

$$W(\nu_x, \nu_y, \nu_t) = \iiint_{-\infty}^{\infty} C(x, y, t) \exp[i2\pi(\nu_x x + \nu_y y + \nu_t t)]\, dx\, dy\, dt. \quad (3)$$

With isotropic media C is spatially a function only of

$$r = \sqrt{(x^2 + y^2)}$$

rather than of x and y individually. The integrals (1), (2) and (3) then reduce to double integrals†. The isotropic equivalents of these integrals, marked by a caret, are then

$$\hat{T}(\nu_r, \nu_t) = 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} \hat{P}(r, t) r J_0(2\pi\nu_r r) \exp(i2\pi\nu_t t)\, dr\, dt, \quad (1')$$

$$C(r, 0, t) = \lim_{R, T \to \infty} \frac{2}{R^2 T} \int_0^T \int_0^R r s(r', 0, t') s(r'+r, 0, t'+t)\, dr'\, dt', \quad (2')$$

$$\hat{W}(\nu_r, \nu_t) = 2\pi \int \int_0^{\infty} r C(r, 0, t) J_0(2\pi\nu_r r) \exp(i2\pi\nu_t t)\, dr\, dt. \quad (3')$$

Thus we may calculate the Wiener spectrum from a measurement of the auto-correlation function C. The Wiener spectrum, in turn, permits us to predict the results of other measurements as illustrated in subsequent sections.

Clearly an important convenience results if the auto-correlation function can be written as the product of two factors, each a function of just one variable:

$$C(x, y, t) = C_1(r) \,.\, C_2(t). \quad (4)$$

This condition implies that an auto-correlation measured with a fixed time delay introduced between the two signal measurements is similar to that taken at any other time delay, i.e.

$$C(r, 0, t_1) = k(t_1 - t_2) C(r, 0, t_2), \quad (5)$$

where k is independent of the independent variable r.

† Note that, with circular symmetry $[f(x, y) = \hat{f}(r)]$:

$$F(\nu_x, \nu_y) = \int \int_{-\infty}^{\infty} f(x, y) \exp[i2\pi(\nu_x x + \nu_y y)]\, dx\, dy$$

can be written [2]

$$\hat{F}(\nu_r) = 2\pi \int_0^{\infty} \hat{f}(r) J_0(2\pi\nu_r r)\, r\, dr,$$

where also

[illegible]

and

[illegible]

The *measured* Wiener spectrum is then [cf. (8) and (10)]

$$\begin{aligned} W_{nm} &= W_n A^2 \\ &= \left[\frac{2J_1(2\pi\nu_r R)}{2\pi\nu_r R}\cdot\frac{\sin 2\pi\nu_t T}{2\pi\nu_t T}\right]^2 \mathrm{W}_n \end{aligned} \tag{12}$$

for the three-dimensional 'top hat' distribution discussed earlier.

Recall that the auto-correlation is the inverse Fourier transform of the Wiener spectrum:

$$C(x, y, t) = s \odot s = \mathscr{F}^{-1}[W] \tag{13}$$

and that the variance of s is

$$\operatorname{var}(s) \equiv \sigma^2(s) \equiv C(0, 0, 0) = \mathscr{F}^{-1}[W]_0 = 2\pi \int_0^\infty \int_{-\infty}^\infty \nu_r W \, d\nu_r \, d\nu_t. \tag{14}$$

Thus the observed variance, i.e. the mean squared value of noise will be:

$$\operatorname{var}(s_n) = 2\pi \int_0^\infty \int_{-\infty}^\infty \nu_r A^2 W_n \, d\nu_r \, d\nu_t. \tag{15}$$

If the variance is measured as R and T are varied, we obtain an integral equation for W_n, which may not be easy to solve.

It is, however, easy to predict the measured variance values if the noise spectrum is white. We have then, with our 'top hat' aperture function [3]

$$\begin{aligned} \operatorname{var}(s_n)_{R,T} &= 2\pi W_n \int_0^\infty \nu_r \left[\frac{2J_1(2\pi\nu_r R)}{2\pi\nu_r R}\right]^2 d\nu_r \int_{-\infty}^\infty \left[\frac{\sin 2\pi\nu_t T}{2\pi\nu_t T}\right]^2 d\nu_t \\ &= W_n/(\pi R^2)(2T). \end{aligned} \tag{16}$$

Thus, if the r.m.s. fluctuation is found to vary inversely with area and interval, this is a good indication that the noise spectrum is white—down to frequencies high compared to the inverse distances and intervals used in the measurements.

Acknowledgments

This work was supported by the Naval Applied Science Laboratory, Brooklyn, N.Y. The encouragement of Mr. M. Halberstam and Mr. Frank Gregory is appreciated.

Certains systèmes optiques qui forment des images présentent une réponse qui varie avec les fréquences temporelles et spatiales. On présente ici des méthodes pour traiter de tels systèmes et on applique celles-ci à des exemples simples. On introduit les notions de réponse ponctuelle-impulsionnelle et de fonction de transfert optique spatio-temporel.

Es gibt abbildende optische Systeme, deren Übertragungseigenschaften sich sowoh mit der örtlichen als auch mit der zeitlichen Frequenz ändern. In der Arbeit werden Verfahren zur Behandlung solcher Systeme dargelegt und auf einfache Beispiele angewandt. Dazu werden zwei Begriffe benutzt : der einer Punktimpuls-Wiedergabe und der optischen Übertragungs einer raumzeitlichen funktion.

References

[1] Beran, M. J., and Parrent, G. B., 1964, *Theory of Partial Coherence* (Englewood Cliffs : Prentice-Hall), § 7.1.

[2] O'Neill, E. L., 1963, *Introduction to Statistical Optics* (Reading : Addison-Wesley), Appendix A.

[3] Gradshteyn, I. S., and Ryzhik, I. M., 1965, *Tables of Integrals, Series and Products* (NewYork : Academic Press), Formulae Nos. 6.538.2 and 3.821.9.