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Equivalent passband of perfect lens

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The equivalent passband of an optical imaging system is defined as the total integral of the square of the modulation transfer function.[1] It has also been called the "structural resolution."[2] It is an important performance criterion because it measures the system's ability to pass signal power, for an input signal having a white spatial-power spectrum. Also, the noise power passed by an imaging system is directly proportional to the equivalent passband when the image noise is white—for instance, if the image consists of randomly spaced small dots, with their concentration representing the signal (e.g., fluoroscopic, radiographic, or photographic images).

When evaluating lenses, we often wish to determine how closely they approach an aberrationless lens, whose modulation transfer function is controlled by diffraction effects exclusively. The ratio of the equivalent passband of the lens to that of an aberrationless lens is therefore a useful criterion. Hence, we wish to know the passband of a perfect lens. Because it does not seem to have been evaluated in closed form, we give the result here.

The modulation transfer function of an aberrationless lens with circular aperture is given by[3]

$$\begin{aligned} F(y) &= \frac{2}{\pi}(\cos^{-1}y - y[1-y^2]^{\frac{1}{2}}), \quad y<1 \\ &= 0, \qquad\qquad y>1; \quad y=\lambda\nu/2A \end{aligned} \tag{1}$$

where λ is the (*in vacuo*) wavelength, ν is the spatial frequency, and A is the numerical aperture of the lens.

The value of ν for which y equals unity is the cut-off frequency beyond which F vanishes.

We evaluate the integral

$$\begin{aligned} I &= \int_0^1 F^2(y)\, dy \\ &= \frac{4}{\pi^2}(A+B+C), \end{aligned} \tag{2}$$

where[4]

$$\begin{aligned} A &= \int_0^1 (\cos^{-1}y)^2\, dy = \{y(\cos^{-1}y)^2 \\ &\qquad -2y - 2[1-y^2]^{\frac{1}{2}}\cos^{-1}y\}_0{}^1 \\ &= \pi - 2, \end{aligned} \tag{3}$$

$$B = -2\int_0^1 y[1-y^2]^{\frac{1}{2}} \cos^{-1}y\, dy.$$

We substitute $\cos^{-1}y = w$. Then

$$B = -2\int_0^{\pi/2} w\sin^2 w \cos w\, dw.$$

When integrated by parts, this yields

$$\begin{aligned} B &= -\tfrac{2}{3}[w\sin^3 w - \tfrac{1}{3}\cos^3 w + \cos w]_0{}^{\pi/2} \\ &= -\pi/3 + 4/9, \end{aligned} \tag{4}$$

$$C = \int_0^1 y^2(1-y^2)dy = 2/15. \tag{5}$$

Hence, substituting Eqs. (3)–(5) into (2), we obtain

$$I = 8(15\pi - 32)/45\pi^2 = 0.2724214\ldots.$$

This value agrees perfectly with the value previously obtained by numerical integration.[5]

[1]O. H. Schade in *Optical Image Evaluation*, Natl. Bur. Stand. (U. S.) Circ. No. 256 (U. S. Government Printing Office, Washington, D. C., 1954), p. 231.
[2]E. H. Linfoot, Opt. Acta 4, 12 (1957).
[3]E. L. O'Neill, *Introduction to Statistical Optics* (Addison-Wesley, Reading, Mass., 1963), p. 84.
[4]H. B. Dwight, *Tables of Integrals and Other Mathematical Functions, 4th ed.* (Macmillan, New York, 1961), Eq. (251).
[5]L. Levi and R. H. Austing, Appl. Opt. 7, 967 (1968).