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DUAL BOOTSTRAP IN CLUSTER MODELS

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The t-dependent dual bootstrap scheme is studied in various cluster models in the j-plane. The importance of the analytic structure of the input amplitudes and proper counting of the contribution of intermediate states to the unitarity sum is shown. Certain counting procedures that correlate clusters and gaps in rapidity lead to integral equations which are not of the usual Chew–Goldberger–Low type. It is these counting procedures that may lead to a self-consistent bootstrap scheme (i.e. no cuts in the output amplitude). Some problems related to the roles of the reggeon-reggeon amplitudes, both in the planar bootstrap and in the calculation of the pomeron, are briefly discussed.

1. Introduction

Much progress has been attained in the dual bootstrap programme [1–9]. Reasonable values for various dynamical quantities (i.e. reggeon intercepts and slopes, couplings) have been calculated analytically or numerically. The purpose of the programme is to implement (i) consistency between the input and output Regge poles (i.e. a bootstrap without cuts [8]) constrained by planar unitarity; (ii) duality constraints on production amplitudes, in particular, finite-energy sum-rules (FESR); (iii) 'clusters' with a variable mass spectrum or equivalently, a variable extent in rapidity; (iv) no double counting conditions (NDC) of either a dynamical or a kinematic origin; (v) reasonable t-dependence and structure for triple-reggeon couplings. The hope is that the input with the constraints mentioned here may be sufficient to allow the determination of the essential features of scattering amplitudes including the pomeron contribution.

As for the j-plane content of the bootstrap scheme, there are still open questions regarding the inclusion of clusters of finite extent in rapidity and NDC. Until now, the only known consistent (i.e. no cuts) j-plane model has been constructed with zero rapidity extent clusters [8] (for particle-particle and reggeon-particle amplitudes).

In this paper we focus attention on the issues of NDC and cluster size. Working within the framework of the dual bootstrap programme, we use multi-Regge kine-

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matics in rapidity variables, factorizable exchanges and FESR over clusters. Only s-channel constraints are studied. Internal quantum numbers are not discussed. We find that the choice of NDC crucially affects the nature of the resulting integral equations and hence also the consequences of the bootstrap scheme. Some common choices of NDC (which may lead to the Chew–Goldberger–Low type [10] equations) cannot meet all the consistency criteria; others may do so in very restrictive cases. We introduce a new NDC (leading to integral equations *not* of the Chew–Goldberger–Low type [10]) that leads, for *arbitrary* cluster size cut-off, to an internally consistent scheme namely, pure Regge pole to Regge pole bootstrap with no cuts introduced at the planar level. This NDC correlates clusters to adjacent reggeon exchanges in a simple but non-trivial manner. The condition that determines the position of the Regge pole coincides with the Rosenzweig–Veneziano condition [5]. Some ambiguities that may occur in calculations with only zero size clusters are resolved. For other choices of NDC, self-consistency seems unlikely and the bootstrap condition on the triple-Regge couplings depends on the cluster size cut-off.

In this paper we study only the J-plane structure of amplitudes. A detailed study in the energy plane will be published elsewhere.

In sect. 2 we describe the general dynamical picture and established the kinematic approximations, analytic assumptions and notation.

In sect. 3 we briefly illustrate that some of the basic problems discussed later on are encountered already in the one dimensional case (no t-dependence). Similar problems are encountered in t-dependent bootstraps with NDC that do not explicitly correlate clusters and adjacent reggeon exchanges. As an example, we discuss in appendix A the NDC of maximal cluster size ($\bar{L}$) and minimal rapidity gap ($\bar{g} \geqslant \bar{L}$) between clusters. In this case, unless particular ad hoc assumptions are imposed, the Rosenzweig–Veneziano condition [5] is not obtained, cuts cannot be killed at the planar level, and the usual one-dimensional limit [11,12] is not attained. Furthermore, these *ad hoc* assumptions are incompatible with the usual expectation for simple pole dominated amplitudes.

Sect. 4 deals with the simplest NDC with gap-cluster correlations, and contains the main results of this paper. In this section, we consider a NDC which correlates clusters to adjacent gaps in an asymmetric manner. For arbitrary cluster size cut-off, the resulting bootstrap condition is that of Rosenzweig and Veneziano [5]. Cut-killing is readily obtained.

The need for caution in certain calculations involving reggeon-reggeon amplitudes is motivated in appendix B. We also leave to appendix C, some details regarding a symmetric NDC correlating gaps and clusters. We find that the desired factorization properties of the output amplitudes are easily generated by the unitarity equations despite the correlations between adjacent clusters. Although the overall consistency (i.e. no cuts) remains an open problem, there is in general an $\bar{L}$-dependent bootstrap condition on the couplings. For $\bar{L} = 0$ overall consistency is achieved. We show how sensitive the bootstrap scheme is to deviations from the proper analyticity assumptions for the input amplitudes.

Sect. 5 is devoted to a few remarks related to the conventional [2] calculation of the pomeron amplitude.

We conclude in sect. 6 with a few summarizing remarks.

2. Dynamical picture, kinematics and notation

As is usual in the dual bootstrap scheme [1,2], we start with the planar s-channel unitarity relation for the reggeon part of the elastic scattering amplitude:

$$\operatorname{Im} A_{\mathrm{ab}\to\mathrm{a'b'}}(s,t) = \sum_n \int \mathrm{d}\Omega_n\, A_{\mathrm{ab}\to n}(\Omega_n)\, A^*_{\mathrm{a'b'}\to n}(\Omega_n)\,. \tag{2.1}$$

The summation is over all possible intermediate states n; $A_{\mathrm{ab}\to n}$ denotes the planar component of the production amplitude. Multiparticle amplitudes are "sewn" together in a planar fashion. Ω_n denotes the intermediate state phase space. Internal quantum numbers are not considered. We want to impose duality constraints on the amplitudes in (2.1) in terms of finite-energy [13] or finite-mass sum rules [14] (FESR, FMSR). For the left-hand side of eq. (2.1), this requires

$$\int_{\mathrm{th}}^{\bar{s}} \mathrm{d}s\, s^K \operatorname{Im} A_{\mathrm{ab}\to\mathrm{a'b'}}(s,t) \simeq \sum_i \frac{\bar{s}^{\alpha_i(t)+K+1}}{\alpha_i(t)+K+1} F^i_{\mathrm{b}}(t) F^i_{\mathrm{a}}(t)\,, \tag{2.2}$$

where $\bar{s}$ is sufficiently large so that, for $s > \bar{s}$, a simple Regge-pole description of the amplitude is a good approximation; $\alpha_i(t)$ are the corresponding Regge-pole trajectories and $F^i_{\mathrm{b}}(t), F^i_{\mathrm{a}}(t)$ are the couplings to the external particles.

For the application of duality constraints to production amplitudes, it is convenient to describe the intermediate state in terms of the longitudinal rapidities * of the particles and momentum transfers. The total available rapidity is $Y \simeq \ln(s)$ and the particles are arranged along the rapidity axis. In order to average over finite ranges in rapidity (i.e. over finite subenergies in the intermediate state) we choose a rapidity cut-off $\bar{L}$, and divide the intermediate state in a given event into groups of particles occupying rapidity ranges (from now on called clusters) with sizes ranging up to $\bar{L}$. One *possible* division (which corresponds to the model discussed in sect. 4) is the following one (see fig. 1): start from one edge of the rapidity axis (say, b). Include in the first cluster as many particles as possible such that the distance between the last particle r_0 in the cluster and b does not exceed $\bar{L}$, but the distance between the next particle l_1 along the chain and b does exceed $\bar{L}$. The size of the first cluster is denoted by L_0. The second cluster begins with l_1 and is defined in a similar manner. Its size is denoted by L_1. Across the non-negative gap g_1 between the two clusters, we assume reggeon exchange. Carry out this division along the chain until the other

* The whole formalism can also be described in terms of invariant mass variables. Only the multi-Regge approximation is essential.

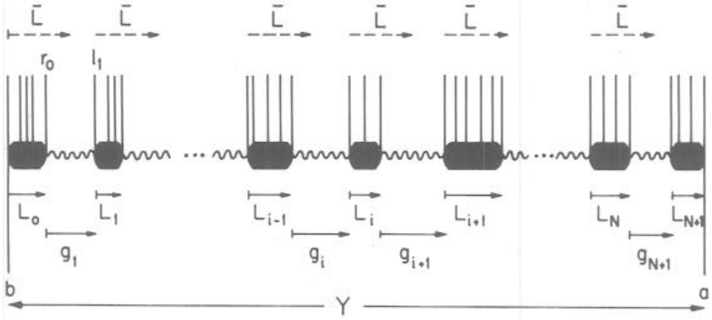

Fig. 1. Grouping into clusters of the intermediate state particle configurations. This particular grouping corresponds to the asymmetric NDC of sect. 4.

edge of the rapidity axis is met. In principle, a single Regge exchange across a gap is a good description only when the gap is large. For small gaps, in a dual theory, one should include an infinite number of poles. We adhere to the single pole approximation since it turns out that including more poles does not alter the conclusions.

The phase-space integration over particles within each cluster is performed and the r.h.s. of eq. (2.1) is described in terms of the cluster phase space and the production amplitudes for $N + 2$ clusters. The resulting unitarity equation is depicted in fig. 2, where to each gap g_i we attach the corresponding momentum transfers t_i^+, t_i^-. Apart from the Regge exchanges across gaps, we are left with the discontinuity across the cluster production amplitudes.

Note that when summing over all particle configurations in all events, one should avoid double counting in terms of the cluster contributions. Therefore, the $N + 2$ cluster phase space must explicitly include a no-double-counting condition (NDC). The NDC should guarantee that the contribution of two adjacent clusters is not confused with that of a single cluster over the same rapidity range.

Assuming multi-Regge kinematics, the $N + 2$ cluster phase space becomes:

$$\mathrm{d}\Omega_{N+2}^{(\mathrm{clusters})} = \mathrm{e}^{-Y} \left[\prod_{i=0}^{N+1} \mathrm{d}L_i \, \mathrm{e}^{L_i} \, \theta(\bar{L} - L_i) \, \theta(L_i) \right] \left[\prod_{i=1}^{N+1} \mathrm{d}g_i \, \theta(g_i) \right] \left[\prod_{i=1}^{N+1} \mathrm{d}\phi_i^{\pm} \right]$$

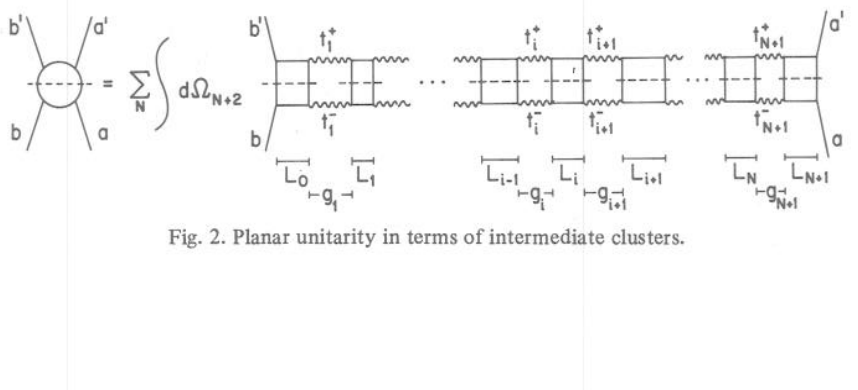

Fig. 2. Planar unitarity in terms of intermediate clusters.

$$\times\, \delta(Y - \sum_{i=0}^{N+1} L_i - \sum_{i=1}^{N+1} g_i)[\mathrm{NDC}]\,. \tag{2.3}$$

Here,

$$\mathrm{d}\phi_i^{\pm} = \frac{\mathrm{d}t_i^+ \mathrm{d}t_i^- \,\theta(-\lambda(t_i^+, t_i^-, t))}{\pi\sqrt{-\lambda(t_i^+, t_i^-, t)}}\,,$$

$$\lambda(t_i^+, t_i^-, t) = t^2 + t_i^{+2} + t_i^{-2} - 2tt_i^+ - 2tt_i^- - 2t_i^+ t_i^-\,,$$

$$[\mathrm{NDC}] = \prod_{i=1}^{N+1} \theta(L_{i-1} + g_i - \bar{L})\,. \tag{2.4}$$

Notice that this [NDC] factor corresponds to the counting of events discussed earlier.

In the preceding discussion the cut-off $\bar{L}$ is used for the averaging over low sub-energies. It may be viewed as the analogue to the usual cut-off in two-body FESR (eq. (2.2)). Therefore, $\bar{L}$ need not have an inherent dynamical significance and may serve as a mere bookkeeping device. The [NDC] factor is part of the definition of phase space, as in eq. (2.3). One may then expect the bootstrap constraints to be independent of $\bar{L}$.

On the other hand, $\bar{L}$ might have some physical significance in certain theoretical models where there is a dynamical significance to the intermediate "cluster states". For example, if hadronic production is assumed to proceed *via* narrow resonances with Regge exchanges, then $\bar{L}$ should be related to the range within which the narrow resonance approximation is valid [2]. As another example, suppose one views cluster production amplitudes (fig. 2) as duality diagrams with quark lines. If the latter are interpreted as actual quarks (partons), the 'clusters' are $q\bar{q}$ pairs. In this context, $\bar{L}$ represents the maximum extent in rapidity of a $q\bar{q}$ pair stable against vacuum polarization. Once the separation between a quark and anti-quark exceeds $\bar{L}$, quark confining forces cause the filling in of their separation with additional $q\bar{q}$ pairs [15]. $\bar{L}$ will affect the $q\bar{q}$ pair multiplicity [16] (i.e. the distribution of partons in the sea). Here one expects an $\bar{L}$ dependent bootstrap condition.

In models with "dynamical clusters" certain constraints on input cluster amplitudes usually occur. Although they are of a different nature from that of the kinematic NDC discussed earlier, we shall denote all such constraints by the same label, NDC (effectively, the constraints we discuss are multiplicative factors in the unitarity integral).

Let us list in order of complexity the NDC to be studied here, with reference to fig. 2 where the unitarity equation is represented in terms of clusters (which may be of either a dynamical or a kinematic nature):

(i) fixed size clusters ($L_i = \bar{L}$) without overlap, sect. 3; (2.5a)

(ii) maximal cluster size ($L_i \leqslant \bar{L}$) and minimal gap size ($g_i \geqslant \bar{L}$), appendix A; (2.5b)

(iii) asymmetric NDC: $L_i \leqslant \bar{L}$ and $L_{i-1} + g_i \geqslant \bar{L}$ (or alternatively, $g_i + L_i \geqslant \bar{L}$), sect. 4; (2.5c)

(iv) symmetric NDC: $L_i \leqslant \bar{L}$ and $L_{i-1} + g_i + L_i \geqslant \bar{L}$, appendix C. (2.5d)

Of the four NDC mentioned here, only condition (iii) corresponds to the proper counting of events as discussed earlier. Conditions (i) and (ii), if interpreted as phase space counting procedures do avoid double counting at the expense of leaving out some allowed configurations. These two constraints correspond to models in which threshold effects are explicitly imposed on input amplitudes. Condition (iv) does not avoid double counting. At the equality point ($L_{i-1} + g_i + L_i = \bar{L}$) two clusters can be confused with a single one. This NDC has been proposed [2] as an a proximate way of implementing no double counting in the duality sense.

Thus, conditions (i), (ii) and (iv) correspond to dynamical pictures in which the missing or doubly counted configurations are presumed to be negligible. Condition (iii), which corresponds to the assignment of clusters discussed earlier (see fig. 1) counts all phase space configurations once and only once. In the spirit of multi-Regge kinematics, we neglect $(t_i^{\pm})_{\min}$ effects explicitly; however, most NDC that we impose involve some repulsion between clusters. $(t_i^{\pm})_{\min}$ effects are of a similar nature and so may be partially accounted for.

In order to formulate in the J plane the bootstrap on the amplitudes shown in fig. 2, we perform the Mellin transform

$$\int_0^{\infty} \mathrm{d}Y\, \mathrm{e}^{-JY} \operatorname{Im} A_{\mathrm{ab}\to\mathrm{a'b'}}(Y, t) \equiv \langle b|\tilde{A}(J, t)|a\rangle\,. \tag{2.6}$$

The effect of the Mellin transformation on the cluster phase space integration (see eq. (2.3)) gives a factorized (*modulo the NDC*) product of transforms over cluster lengths and gaps. The shape of the resulting integral equation will depend on just how the NDC, the FMSR for cluster four-point functions (or factorization properties of the Mellin-transforms over clusters) and the transforms over the reggeon exchanges across gaps combine.

Among the quantities we shall encounter are transforms of the reggeons exchanged across gaps. These reggeons are assumed to be simple poles so that

$$\int_{\bar{g}}^{\infty} \mathrm{d}g\, \mathrm{e}^{-g(J+1)} X(g, t^{\pm}) = \tilde{X}(J, t^{\pm})\, \mathrm{e}^{-\bar{g}(J-\alpha_{\mathrm{c}}(t^{\pm}))}\,, \tag{2.7}$$

where $X(g, t^{\pm})$ is the function describing the two-reggeon propagator and

$$\alpha_{\mathrm{c}}(t^{\pm}) \equiv \alpha(t^{+}) + \alpha(t^{-}) - 1\,,$$

$$\widetilde{X}(J, t^{\pm}) \equiv \cos[\pi(\alpha(t^{+}) - \alpha(t^{-}))]/(J - \alpha_c(t^{\pm})) \,. \tag{2.8}$$

The (cut-off) Mellin transforms over the discontinuity of cluster four-point functions in the input amplitudes are denoted

$$\langle t_l^{\pm} | \widetilde{G}^{\bar{L}}(J, t) | t_r^{\pm} \rangle \equiv \int_0^{\bar{L}} \mathrm{d}L \, \mathrm{e}^{-JL} \langle t_l^{\pm} | G(L, t) | t_r^{\pm} \rangle \,, \tag{2.9}$$

where $t_l^{\pm}$, $t_r^{\pm}$ display the dependence on the reggeon legs attached to a cluster. For end clusters, the appropriate legs are replaced by the particles. The triple-reggeon couplings buried in such transforms will be denoted by $F(t^{\pm}, t)$. We do not have at hand the explicit analytic structure of the input amplitudes. Instead, we demand for the full (Mellin-transformed) output amplitude $\langle l | \widetilde{A}(J, t) | r \rangle$ and the Born term $\langle l | \widetilde{A}^{\bar{L}}(J, t) | r \rangle$, the following relation which follows from the simple pole dominance of the amplitude (and hence is also implicit in FESR and FMSR):

$$\langle l | \widetilde{A}(J, t) | r \rangle - \langle l | \widetilde{A}^{\bar{L}}(J, t) | r \rangle = \frac{F_l(t) F_r(t) \, \mathrm{e}^{-\bar{L}(J - \alpha(t))}}{J - \alpha(t)} \,. \tag{2.10}$$

Here l and r may be external particle or reggeon legs. Whenever FMSR are encountered we shall make explicit assumptions on their form.

Let us stress at the outset that because of the simplicity of our approach some of the possible complexity of various Mellin transforms over four-reggeon amplitudes is not apparent.

We shall see that some of the NDC (2.5) have the same one-dimensional limit as the original Huan–Lee, Veneziano bootstrap [11,12]. NDC (2.5c) also naturally generates the Rosenzweig–Veneziano bootstrap condition [5]:

$$1 = \int \mathrm{d}\phi^{\pm} \, \frac{F^2(t^{\pm}, t) \, \widetilde{X}(\alpha(t), t^{\pm})}{\alpha(t) - \alpha_c(t^{\pm})} \,. \tag{2.11}$$

3. Self-consistency in models with no gap-cluster correlations

Consider the simple example of a one-dimensional (i.e. no t dependence) model having $\bar{L} = 0$ [11,12]. In calculating the unitarity sum of fig. 2 one substitutes for the input exchanges and cluster amplitudes

$$\widetilde{X}(J, t^{\pm}) \to \frac{1}{J - \alpha_c} \,, \qquad \alpha_c = 2\alpha - 1 \,,$$

$$\langle t_l^{\pm} | \widetilde{G}^{\bar{L}}(J, 0) | t_r^{\pm} \rangle \to g^2 s \,, \qquad \langle b | \widetilde{G}^{\bar{L}}(J, 0) | a \rangle \to g^2 s_1 \,,$$

$$\langle t_l^\pm | \widetilde{G}^{\overline{L}}(J,0)|a\rangle \to g^2 s_2 \,. \tag{3.1}$$

One then finds for the output amplitudes:

$$\langle t_l^\pm | \widetilde{A}(J,0)|t_r^\pm\rangle = \frac{g^2 s(J-\alpha_c)}{J-\alpha_c - g^2 s}\,, \qquad \text{(reggeon-reggeon)}\,, \tag{3.2a}$$

$$\langle t_l^\pm | \widetilde{A}(J,0)|a\rangle = \frac{g^2 s_2(J-\alpha_c)}{J-\alpha_2 - g^2 s}\,, \qquad \text{(reggeon-particle)}\,, \tag{3.2b}$$

$$\langle b | \widetilde{A}(J,0)|a\rangle = g^2 s_1 + \frac{g^4 s_2^2}{J-\alpha_c - g^2 s}\,, \qquad \text{(particle-particle)}\,. \tag{3.2c}$$

The bootstrap constraint requires that at $J = \alpha$, these amplitudes have a simple pole with a residue equal to g^2. This gives

$$g^2 = (\alpha - \alpha_c)^2 = (1-\alpha)^2\,, \qquad s = s_2 = \frac{1}{\alpha - \alpha_c} = \frac{1}{1-\alpha}\,, \tag{3.3}$$

so that

$$\widetilde{A}_R(J) \equiv \langle t_l^\pm | \widetilde{A}(J,0)|t_r^\pm\rangle = \langle t_l^\pm | \widetilde{A}(J,0)|a\rangle = \frac{g^2}{(J-\alpha)}\frac{(J-\alpha_c)}{(\alpha-\alpha_c)}\,, \tag{3.4a}$$

$$\langle b | \widetilde{A}(J,0)|a\rangle = g^2 s_1 + \frac{g^2}{J-\alpha}. \tag{3.4b}$$

Note that complete consistency necessitates the inclusion of the 'finite-energy sum-rule' factors s, s_1, $s_2 \neq 1$ for $\alpha \simeq \frac{1}{2}$. The full amplitudes have not just a simple pole but also include some smooth terms. It is important to notice that the FMSR factors s, s_2 depend on the reggeon legs attached to the clusters.

In the usual construction of the cylinder (bare pomeron) through iterating reggeon amplitudes and twisted propagators [2] (as in fig. 3), it is the full amplitudes (eq. (3.4a)) that must be used to get the simple Huan–Lee, Veneziano [11,12] result $\alpha_P = 1$ for the pomeron intercept from the relation

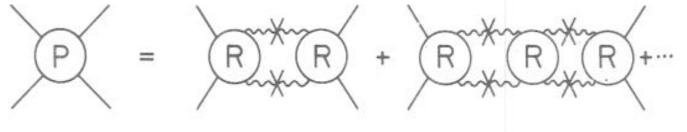

Fig. 3. Conventional calculation of the pomeron by the iteration of planar amplitudes.

$$\langle t_l^\pm | \widetilde{A}_P(J,0) | t_r^\pm \rangle = \frac{\widetilde{A}_R^2(J)/(J-\alpha_c)}{1-\widetilde{A}_R(J)/(J-\alpha_c)} . \tag{3.5}$$

The naive substitution $\widetilde{A}_R(J) \simeq g^2/(J-\alpha)$ gives instead [18]

$$(\alpha_P - 1)(\alpha_P - \alpha_c) - g^2 = 0 . \tag{3.6}$$

Here $\alpha_P(0)$ does *not* equal one if g^2 is determined by the reggeon bootstrap condition of eq. (3.3). The only way to get $\alpha_P = 1$ in this case is to give up the bootstrap constraints on g^2! We see that the simple counting of diagrams [11] corresponds to the strict use of the reggeon bootstrap conditions at the pomeron level as well.

In order to demonstrate the degree of complexity involved in averaging over clusters of non-zero maximal size $\bar{L}$, consider the NDC of eq. (2.5a) in a one-dimensional model [17].

In this case, the input entities are:

$$\langle t_l^\pm | \widetilde{G}^{\bar{L}}(J,0) | t_r^\pm \rangle \to g^2\, e^{-J\bar{L}} D , \qquad \widetilde{X}(J, t^\pm) \to \frac{1}{J-\alpha_c} . \tag{3.7}$$

Consistency at the pole of the reggeon-reggeon amplitude requires:

$$\begin{aligned} &Dg^2\, e^{-\alpha\bar{L}}/(\alpha-\alpha_c) = 1 , && \text{(pole position)} , \\ &\frac{g^2 e^{-2\alpha\bar{L}} D^2/(\alpha-\alpha_c)}{\bar{L} + g^2 D\, e^{-\alpha\bar{L}}/(\alpha-\alpha_c)^2} = 1 , && \text{(residue)} . \end{aligned} \tag{3.8}$$

Hence,

$$D = e^{\alpha\bar{L}}[\bar{L} + (\alpha-\alpha_c)^{-1}] , \qquad g^2 = \frac{(\alpha-\alpha_c)^2}{1+\bar{L}(\alpha-\alpha_c)} . \tag{3.9}$$

The the bootstrap conditions depend on $\bar{L}$. This is characteristic of most simple bootstrap schemes with clusters. Only particular NDC may lead to an $\bar{L}$ independent bootstrap.

Using eq. (3.9), we find an output reggeon-reggeon amplitude of the form

$$\widetilde{A}_{RR}(J) = \frac{(\alpha-\alpha_c)(J-\alpha_c)\, e^{-\bar{L}(J-\alpha)}}{J-\alpha_c-(\alpha-\alpha_c)\, e^{-\bar{L}(J-\alpha)}} . \tag{3.10}$$

Notice that although consistency at the pole is satisfied for this particular NDC, the resulting amplitude has additional singularities (pairs of conjugate complex poles [17, 18]) unavoidable in this model. Such complex poles are the analogue of cuts in models with t-dependence.

Both simple models that we have studied in this section suggest that the amplitudes resulting from bootstrap schemes have not only simple poles but also require some additional smooth functions of J. It is also evident that care must be taken when calculating the pomeron.

Rather than beleaguer the reader here with yet another complicated (and unsatisfactory) model, we defer to appendix A the discussion of a t-dependent bootstrap with the NDC (2.5b) imposing a minimum gap $\theta(g_i - \bar{L})$ in addition to a maximum cluster size $\theta(\bar{L} - L_i)$. Internal consistency for this scheme is unlikely (and in certain instances impossible).

4. Gap-cluster correlations *via* NDC

We now consider the NDC of equation (2.5c). That is, for a given cluster of length L_i and one adjacent gap (always to the left or always to the right) one of the following conditions should be satisfied * (see fig. 4a, b)

$$L_{i-1} + g_i \geqslant \bar{L} \tag{4.1a}$$

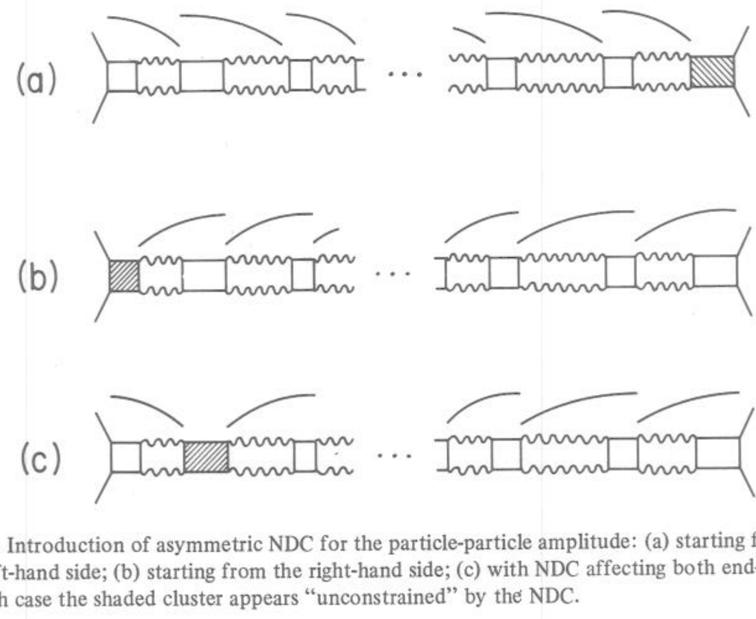

Fig. 4. Introduction of asymmetric NDC for the particle-particle amplitude: (a) starting from the left-hand side; (b) starting from the right-hand side; (c) with NDC affecting both end-clusters. In each case the shaded cluster appears "unconstrained" by the NDC.

* After completion of this work, we learned of the related work by Finkelstein and Koplik [19] for a simple one dimensional case. Our input analytic assumptions (i.e. FMSR) differ significantly from theirs; they assume an explicit cluster spectrum in rapidity that neglects the low mass (i.e. small rapidity extent) behaviour of cluster amplitudes. Thus we do not run into the difficulties that they encounter in achieving consistency.

or

$$L_i + g_i \geqslant \bar{L} \ . \tag{4.1b}$$

Superficially, some double counting seems to occur at the equality points. For example in fig. 2, the two cluster configuration $\{L_0 + g_1 = \bar{L}$ and $L_1 = 0\}$ overlaps with a single cluster of length $\bar{L}$. Although it affects only a zero measure part of the cluster phase space integrations, care is required. Finite contributions to the unitarity integral arise from configurations involving single-particle intermediate states ($L = 0$ in our language).

With NDC (4.1) we shall show how the Rosenzweig–Veneziano bootstrap condition [5] and the Bishari–Veneziano cut-killing mechanism [8] are naturally generated for arbitrary $\bar{L}$ (without *ad hoc* asymmetries required in output amplitudes). Let us apply condition (4.1a) beginning at the leftmost cluster. This leaves the last cluster on the right on a different footing than all the others (see fig. 4a). Undoing a few integrations from the left, we find the following integral expression for the output amplitude:

$$\langle l|\tilde{A}(J,t)|r\rangle - \langle l|\tilde{A}^{\bar{L}}(J,t)|r\rangle = \int \mathrm{d}\phi_1^{\pm} \int_0^{\bar{L}} \mathrm{d}L_0 \mathrm{e}^{-L_0 J} \langle l|G(L_0,t)|t_1^{\pm}\rangle$$

$$\times \int_0^{\infty} \mathrm{d}g_1 \mathrm{e}^{-g_1(J+1)} X(g_1, t_1^{\pm})\, \theta(L_0 + g_1 - \bar{L}) \langle t_1^{\pm}|\tilde{B}(J,t)|r\rangle \ . \tag{4.2}$$

Here l and r can be either external particle ($l \to b$, $r \to a$) or external reggeon ($l \to t_l^{\pm}$, $r \to t_r^{\pm}$) legs. Performing first the g_1 and then the L_0 integrations *, one finds

$$\langle l|\tilde{A}(J,t)|r\rangle - \langle l|\tilde{A}^{\bar{L}}(J,t)|r\rangle = \int \mathrm{d}\phi_1^{\pm} [\int_0^{\bar{L}} \mathrm{d}L_0 \mathrm{e}^{-L_0 \alpha_{c_1}} \langle l|G(L_0,t)|t_1^{\pm}\rangle] \mathrm{e}^{-\bar{L}(J-\alpha_{c_1})}$$

$$\times \tilde{X}(J, t_1^{\pm}) \langle t_1^{\pm}|\tilde{B}(J,t)|r\rangle \ . \tag{4.3}$$

$\langle t_1^{\pm}|\tilde{B}(J,t)|r\rangle$ satisfies the recursion relation

$$\langle t_1^{\pm}|\tilde{B}(J,t)|r\rangle - \langle t_1^{\pm}|\tilde{B}^{\bar{L}}(J,t)|r\rangle = \int \mathrm{d}\phi_2^{\pm} [\int_0^{\bar{L}} \mathrm{d}L_1 \mathrm{e}^{-\alpha_{c_2} L_1} \langle t_1^{\pm}|G(L_1,t)|t_2^{\pm}\rangle] \mathrm{e}^{-\bar{L}(J-\alpha_{c_2})}$$

$$\times \tilde{X}(J, t_2^{\pm}) \langle t_2^{\pm}|\tilde{B}(J,t)|r\rangle \ . \tag{4.4}$$

One might be tempted to identify $\langle t_1^{\pm}|\tilde{B}(J,t)|r\rangle$ with an amplitude that has free external reggeon legs on the left (i.e., the output amplitude $\langle t_1^{\pm}|\tilde{A}(J,t)|r\rangle$), but this

* This is the proper order to easily implement the NDC and also count all possible configurations.

is not justified in general. The "amplitude" $\langle t_1^\pm|\widetilde{B}(J,t)|r\rangle$ is only defined within the context of the unitarity integral, and is affected by the overall energy conservation δ function of eq. (2.3) in an asymmetric manner. This point, which is unimportant as long as one assumes that the input amplitudes are regular at threshold ($L = 0$), is very crucial once proper analyticity is taken into account. As we shall show, the distinction between $\langle t_1^\pm|\widetilde{A}(J,t)|r\rangle$ and $\langle t_1^\pm|\widetilde{B}(J,t)|r\rangle$ may be relaxed in the discussion of reggeon-particle amplitudes with internal consistency still achieved in a restricted model. This is not the case for reggeon-reggeon [26] amplitudes. In appendix B we show how confusion of the two entities leads into inconsistency. However, from (4.3–4) follows the relation

$$\langle t_1^\pm|\widetilde{B}(J,t)|r\rangle = \langle t_1^\pm|\widetilde{A}(J,t)|r\rangle - \langle t_1^\pm|\widetilde{A}^{\overline{L}}(J,t)|r\rangle + \langle t_1^\pm|\widetilde{B}^{\overline{L}}(J,t)|r\rangle\,. \tag{4.5}$$

Notice that with the NDC discussed here there is no need to explicitly separate the two cluster term (as in the case with the NDC discussed in appendices A, C). In eqs. (4.3, 4.4),the cut-off transforms over discontinuities of input cluster amplitudes are converted into FMSR as direct consequence of the NDC.

We assume the FMSR:

$$\int_0^{\overline{L}} \mathrm{d}L_0 \mathrm{e}^{-L_0\alpha_{c_1}} \langle t_1^\pm|G(L_0,t)|a\rangle = \frac{F(t_1^\pm,t)F_a(t)\,\mathrm{e}^{\overline{L}(\alpha-\alpha_{c_1})}}{\alpha-\alpha_{c_1}}\,,$$

$$\int_0^{\overline{L}} \mathrm{d}L_0 \mathrm{e}^{-L_0\alpha_{c_1}} \langle b|G(L_0,t)|t_1^\pm\rangle = \frac{F(t_1^\pm,t)F_b(t)\,\mathrm{e}^{\overline{L}(\alpha-\alpha_{c_1})}}{\alpha-\alpha_{c_1}}\,, \tag{4.6a}$$

$$\int_0^{\overline{L}} \mathrm{d}L_1 \mathrm{e}^{-L_1\alpha_{c_2}} \langle t_1^\pm|G(L_1,t)|t_2^\pm\rangle = F(t_1^\pm,t)\,F(t_2^\pm,t)\,\frac{\mathrm{e}^{\overline{L}(\alpha-\alpha_{c_2})}}{\alpha-\alpha_{c_2}}\,. \tag{4.6b}$$

Eq. (4.6a) is in accord with the usual FMSR properties of reggeon-particle amplitudes [14] in planar dual theories. Our guess for the form in eq. (4.6b) of the relevant FMSR for reggeon-reggeon amplitude seems natural and a confirmation in the context of dual theory will be published elsewhere [26]. Notice that the "asymmetry" in the FMSR stems naturally from the NDC and does not raise the problems encountered wiht the symmetric NDC of appendix A. We stress that the analytic assumptions (4.6a) and (4.6b) specify particular integrals (i.e. FMSR) over amplitudes and *not* the detailed form of the amplitudes themselves (which are rather complicated). Because the amplitudes also satisfy higher, integer moment FMSR it is misleading to approximate the amplitudes by explicit functions that may be consistent only with (4.5a) and (4.5b). If the B_n of dual models is viewed as a prototype of dual amplitudes it is obvious that one particular moment FESR does not reveal the complicated singularity structure of an amplitude.

The assumed form of the r.h.s. of eq. (4.6) needs some clarification. The lower limit of the integral, $L = 0$, corresponds to the threshold of the cluster amplitude. Rapidity is used merely for convenience. In fact, the NDC (4.1) may be expressed in terms of ratios of mass variables, and the integrals in eq. (4.6) should be interpreted as FMSR over discontinuities of amplitudes from threshold (M_{th}^2) up to $\bar{M}^2 \cong M_{\text{th}}^2 e^{\bar{L}}$. It is in this sense that we assume for the FMSR the form they acquire in the Dual model [14,26]. If one insists on interpreting (4.6) as integrals down to $L = 0$, the absence of a lower limit term in the r.h.s. corresponds to a singular behaviour at threshold. This may be considered as a simulation of the singular contribution of single particle intermediate states (in our language a zero size cluster is not empty).

Calculating for example, the J-plane structure of the two-cluster amplitude one finds that *independent* of whether the leg on the right-hand side (r) is a particle or a reggeon the Born term of eq. (4.4) is:

$$\langle t_1^\pm | \tilde{B}^{\bar{L}}(J, t) | r \rangle = \frac{F(t_1^\pm, t) F_r(t)}{J - \alpha} \left\{ \frac{J - \alpha_{c_1}}{\alpha - \alpha_{c_1}} - e^{-\bar{L}(J-\alpha)} \right\} . \tag{4.7}$$

Thus, NDC (4.1) FMSR (4.6) and the overall energy conservation constraint determine the form of $\langle t_1^\pm | \tilde{B}^{\bar{L}}(J, t) | r \rangle$ *irrespective* of the precise form of the free-leg Born terms $\langle l | \tilde{A}^{\bar{L}}(J, t) | r \rangle$ which never enter into the multi-cluster ($n \geqslant 2$) unitarity integrals. The asymmetric form of eq. (4.7) stems from the fact that only the $t_1^\pm$ leg is constrained by the unitarity integral while the r leg is not.

Using eq. (4.6), eqs. (4.3) and (4.4) now become

$$\langle l | \tilde{A}(J, t) | r \rangle - \langle l | \tilde{A}^{\bar{L}}(J, t) | r \rangle$$

$$= F_l(t) \, e^{-\bar{L}(J-\alpha)} \int d\phi_1^\pm \, F(t_1^\pm, t) \frac{\tilde{X}(J, t_1^\pm)}{(\alpha - \alpha_{c_1})} \langle t_1^\pm | \tilde{B}(J, t) | r \rangle , \tag{4.8}$$

$$\langle t_1^\pm | \tilde{B}(J, t) | r \rangle - \langle t_1^\pm | \tilde{B}^{\bar{L}}(J, t) | r \rangle$$

$$= F(t_1^\pm, t) \, e^{-\bar{L}(J-\alpha)} \int d\phi_2^\pm F(t_2^\pm, t) \frac{\tilde{X}(J, t_2^\pm)}{(\alpha - \alpha_{c_2})} \langle t_2^\pm | \tilde{B}(J, t) | r \rangle . \tag{4.9}$$

The resulting solution for $\langle t_1^\pm | \tilde{B}(J, t) | a \rangle$ is:

$$\langle t_1^\pm | \tilde{B}(J, t) | r \rangle - \langle t_1^\pm | \tilde{B}^{\bar{L}}(J, t) | r \rangle$$

$$= e^{-\bar{L}(J-\alpha)} F(t_1^\pm, t) \frac{\int d\phi_2^\pm F(t_2^\pm, t) \dfrac{\tilde{X}(J, t_2^\pm)}{(\alpha - \alpha_{c_2})} \langle t_2^\pm | \tilde{B}^{\bar{L}}(J, t) | r \rangle}{1 - e^{-\bar{L}(J-\alpha)} I(J, t)} , \tag{4.10}$$

where (see eq. (2.8))

$$I(J,t) \equiv \int \mathrm{d}\phi^{\pm} F^2(t^{\pm}, t)\, \frac{\widetilde{X}(J, t^{\pm})}{\alpha - \alpha_c}$$
$$= \int \mathrm{d}\phi^{\pm}\, \frac{F^2(t^{\pm}, t) \cos[\pi(\alpha(t^{+}) - \alpha(t^{-}))]}{(\alpha - \alpha_c)(J - \alpha_c)} \,. \tag{4.11}$$

Inserting (4.10) into (4.8), we find

$$\langle l|\widetilde{A}(J,t)|r\rangle - \langle l|\widetilde{A}^{\overline{L}}(J,t)|r\rangle$$
$$= \mathrm{e}^{-\overline{L}(J-\alpha)} F_l(t)\, \frac{\int \mathrm{d}\phi_1^{\pm} F(t_1^{\pm}, t)\, \dfrac{\widetilde{X}(J, t_1^{\pm})}{\alpha - \alpha_{c_1}} \langle t_1^{\pm}|\widetilde{B}^{\overline{L}}(J,t)|r\rangle}{1 - \mathrm{e}^{-\overline{L}(J-\alpha)} I(J,t)} \,. \tag{4.12}$$

$\langle t_1^{\pm}|\widetilde{B}^{\overline{L}}(J,t)|r\rangle$ is a smooth function of J; therefore the only singularities occurring on the r.h.s. of eq. (4.12) are possible cuts introduced by the loop integrations (recall the form of $\widetilde{X}(J, t^{\pm})$ given in eq. (2.8)) and poles at zeros of the denominator.

The bootstrap condition for the pole position at $J = \alpha$ becomes

$$I(J = \alpha, t) = 1 \,. \tag{4.13}$$

This is identical to the Rosenzweig–Veneziano condition [5] (see eq. (2.11)). Notice that the only $\overline{L}$ dependence in the denominator of eq. (4.12) occurs in the factor $\mathrm{e}^{-\overline{L}(J-\alpha)}$. Therefore the bootstrap condition (4.13) holds independent of $\overline{L}$!

Full consistency requires in addition, both residue matching at $J = \alpha$ and also the absence of cuts. We now show that both are automatically satisfied. Using eq. (4.13) in eq. (4.12) gives

$$\langle l|\widetilde{A}(J,t)|r\rangle - \langle l|\widetilde{A}^{\overline{L}}(J,t)|r\rangle$$
$$= \frac{F_l(t)\, \mathrm{e}^{-\overline{L}(J-\alpha)} \int \mathrm{d}\phi_1^{\pm} F(t_1^{\pm}, t)\, \dfrac{\widetilde{X}(J, t_1^{\pm})}{\alpha - \alpha_{c_1}} \langle t_1^{\pm}|\widetilde{B}^{\overline{L}}(J,t)|r\rangle}{\int \mathrm{d}\phi^{\pm} F(t^{\pm}, t)\, \dfrac{\widetilde{X}(J, t^{\pm})}{\alpha - \alpha_c} F(t^{\pm}, t) \left[\dfrac{\widetilde{X}(\alpha, t^{\pm})}{\widetilde{X}(J, t^{\pm})} - \mathrm{e}^{-\overline{L}(J-\alpha)}\right]} \,. \tag{4.14}$$

Inserting eq. (4.7) into (4.14) we find

$$\langle l|\widetilde{A}(J,t)|r\rangle - \langle l|\widetilde{A}^{\overline{L}}(J,t)|r\rangle = \frac{F_l(t) F_r(t)}{J - \alpha}\, \mathrm{e}^{-\overline{L}(J-\alpha)} \,. \tag{4.15}$$

Thus, eq. (2.10) is satisfied. Hence with NDC (4.1), FMSR and condition (4.13) it is possible to obtain an output amplitude which has a simple pole, with the correct residue, no cuts and only some possible smooth J dependence (buried in the Born term $\langle l|\widetilde{A}^{L}(J,t)|r\rangle$). This is true regardless of whether the external legs (l, r) are reggeons or particles. With the aid of eq. (4.7) the form of $\langle t_1^\pm|\widetilde{B}(J,t)|r\rangle$ (eq. (4.10)) becomes

$$\langle t_1^\pm|\widetilde{B}(J,t)|r\rangle = \frac{F(t_1^\pm,t)F_r(t)}{J-\alpha}\left[\frac{J-\alpha_{c_1}}{\alpha-\alpha_{c_1}}\right]. \tag{4.16}$$

This is the form of the *output reggeon-particle amplitude* suggested by Bishari and Veneziano [8]. At the same time, the $\overline{L}=0$ limit of eq. (4.7) is

$$\lim_{\overline{L}\to 0}\langle t_1^\pm|\widetilde{B}^{\overline{L}}(J,t)|r\rangle = \frac{F(t_1^\pm,t)F_r(t)}{\alpha-\alpha_{c_1}}, \tag{4.17}$$

which is the cut-off Mellin transform suggested by ref. [8]. The simple form of eq. (4.16) is a *possible* solution for the output reggeon-particle amplitude (for $r\to$ a), which is consistent with the FMSR (4.6a) but not with any higher moments (these should be satisfied as well). As long as one does not worry about the full analytic structure of amplitudes (i.e., higher moment FMSR) one may equate $\langle t_1^\pm|\widetilde{B}^{\overline{L}}(J,t)|a\rangle$ and $\langle t_1^\pm|\widetilde{A}^{\overline{L}}(J,t)|a\rangle$ and not run into inconsistencies at the particle-particle and the reggeon-particle level. Obviously, the resulting integral equations become simpler (eq. (4.8) with $\widetilde{B}$ replaced by $\widetilde{A}$). Notice that even these simpler equations are *not* of a Chew–Goldberger–Low type [10]. However, the need for a distinction between $\langle t_l^\pm|\widetilde{A}^{\overline{L}}(J,t)|t_r^\pm\rangle$ and $\langle t_l^\pm|\widetilde{B}^{\overline{L}}(J,t)|t_r^\pm\rangle$ for reggeon-reggeon amplitudes is evident. The former is symmetric in the external legs while the latter obviously is not. In order to emphasize the importance of this distinction we show in appendix B how abandoning it leads into inconsistencies.

To illustrate the importance of the NDC in models with a cluster cut-off $\overline{L}$, we perform the following trivial exercise. Calculate the particle-particle amplitude from the reggeon-particle amplitude in a "naive" manner yielding a Chew–Goldberger–Low equation [10]:

$$\langle b|\widetilde{A}(J,t)|a\rangle - \langle b|\widetilde{A}^{\overline{L}}(J,t)|a\rangle = \int d\phi_1^\pm \langle b|\widetilde{G}^{\overline{L}}(J,t)|t_1^\pm\rangle\,\widetilde{X}(J,t_1^\pm)\langle t_1^\pm|\widetilde{G}(J,t)|a\rangle. \tag{4.18}$$

Now instead of FMSR (resulting from the NDC) a (unconstrained) cut-off Mellin transform remains.

Using for the reggeon-particle amplitude the *possible* form given by the r.h.s. of eq. (4.16) (this is the form proposed by ref. [8]), and its Born term as given by the r.h.s. of eq. (4.7) we get

$$\langle b|\widetilde{A}(J,t)|a\rangle - \langle b|\widetilde{A}^{\overline{L}}(J,t)|a\rangle$$

$$= \frac{F_b(t)F_a(t)}{J-\alpha}\int d\phi_1^\pm \frac{F^2(t_1^\pm,t)\,\widetilde{X}(\alpha,t_1^\pm)}{J-\alpha}\left[\frac{J-\alpha_{c_1}}{\alpha-\alpha_{c_1}} - e^{-\overline{L}(J-\alpha)}\right]. \tag{4.19}$$

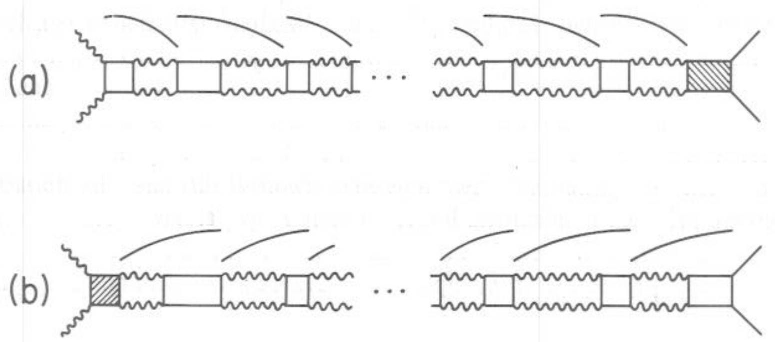

Fig. 5. Introduction of asymmetric NDC for the reggeon-particle amplitude with the (shaded) cluster "unconstrained" by NDC having (a) external particle legs; (b) external reggeon legs.

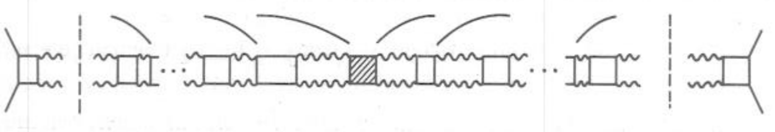

Fig. 6. Isolation of the reggeon-reggeon amplitude.

This obviously violates eq. (2.10) (the assumption of simple pole dominance of the amplitude) except in the $\bar{L} = 0$ limit. Thus models which are consistent only for $\bar{L} = 0$ may be misleading (see also appendix C).

Note that we have calculated in a certain order, namely, the NDC were imposed so that the rightmost cluster was singled out (see figs. 4a and 5a).

It obviously makes no difference if the calculation is done in the reversed order (see, e.g. figs. 4b and 5b); or if cluster assignment is started at both ends of the chain leaving a cluster in the middle singled out (e.g. figs. 4c and 6). However, one must not confuse the contribution of this "left over" cluster (see figs. 4, 5, 6) with that of others. Both the energy conservation factor in eq. (2.3) and the NDC on all other clusters affect this particular cluster. Special attention is required when amplitudes are singular. Of course, dealing with properly defined amplitudes (e.g., in a dual model) is much more complicated. However, the qualitative features (NDC, importance of order of integration over internal variables and cut killing) are not expected to be different.

Let us mention the average cluster multiplicity $\overline{N(Y)}$. Using generating function techniques we find

$$[\overline{N(Y)}/Y] \underset{Y\to\infty}{\longrightarrow} \frac{1}{\bar{L} + \int \mathrm{d}\phi^{\pm} F^2(t_1^{\pm}, t)\, \widetilde{X}(\alpha, t^{\pm})/(\alpha - \alpha_c)^2}\,, \tag{4.20}$$

which is the three-dimensional analogue of one dimensional models [17] where

$$[\overline{N(Y)}/Y] \xrightarrow[Y\to\infty]{} \frac{1}{\bar{L} + 1/(1-\alpha(0))} . \tag{4.21}$$

If the cluster size is a physically significant quantity (if clusters are $q\bar{q}$ pairs, for example) then $\bar{L}$ appears explicitly in various multiplicity moments.

We conclude this section with a few words about two simple symmetric NDC that correlate clusters with adjacent gaps. Suppose both condition (4.1a) and (4.1b) are applied to every cluster:

$$g_i + L_i \geqslant \bar{L} \qquad \text{and} \qquad L_i + g_{i+1} \geqslant \bar{L} . \tag{4.22}$$

This NDC is not fruitful. First of all, it leaves out a certain part of the possible phase space configurations. Moreover it leads to a non-factorizable integral equation.

Some interest may be found in the following symmetric condition [2] (minimal in some sense)

$$L_{i-1} + g_i + L_i \geqslant \bar{L} . \tag{4.23}$$

This NDC allows double counting at the equality points. Condition (4.23) is perhaps natural when the intermediate states of fig. 2 are viewed as narrow resonances [2] or as $q\bar{q}$ pairs. We have examined NDC (4.23) in the one and three dimensional cases *. Some of this study is presented in appendix C. It turns out that, despite the non-trivial correlations between adjacent clusters and gaps, the factorization properties of the output amplitudes are guaranteed by the unitarity equations. The bootstrap condition appears to be $\bar{L}$ dependent. Cut killing for $\bar{L} \neq 0$ remains an open question. For $\bar{L} = 0$ the model is selfconsistent (i.e. no cuts) and the Rosenzweig–Veneziano constraint [5] results (*provided* the proper input analyticity assumptions are made). Replacing the FMSR assumption by an explicit oversimplified form of the cluster amplitudes [19,22] yields an $\bar{L}$ dependent bootstrap condition; however, complete internal consistency (i.e. no cuts) is impossible in this scheme.

5. Problems in calculating the bare pomeron amplitude

The calculation of the bare pomeron amplitude (or, cylinder [1–4]) is still an open problem. One major difficulty is that in the usual calculations, the cylinder amplitude contains, in addition to the leading pomeron pole, a lower-lying pole with a negative residue that cancels the planar reggeon. A possible way out has been proposed recently by Veneziano [23] who examined the demands of t-channel unitarity as well as those of s-channel unitarity.

* After completing this work we learned of related one-dimensional calculations with different input analytic assumptions done by Kwiecinski and Sakai [22] and Finkelstein and Koplik [19].

In this section we adhere to the "conventional" calculation [2] of the cylinder whereby one adds up all chains of reggeon-reggeon amplitudes with all reggeon legs being twisted propagators having no NDC (see fig. 3). We discuss two examples that show how strongly the pomeron singularity depends on the form chosen for the reggeon-reggeon amplitude $\langle t_l | \widetilde{A}(J, t) | t_r \rangle$.

First we consider the choice

$$\langle t_l^{\pm} | \widetilde{A}(J, t) | t_r^{\pm} \rangle = \frac{F(t_l^{\pm}, t) F(t_r^{\pm}, t)}{J - \alpha} \frac{(J - \alpha_{c_l})(J - \alpha_{c_r})(\alpha + 1)}{(\alpha - \alpha_{c_l})(\alpha - \alpha_{c_r})(J + 1)} , \tag{5.1}$$

which continues smoothly into external particle legs to give simple reggeon-particle and particle-particle amplitudes [8].

The total output amplitude (reggeon+pomeron) with external reggeon legs is found to be

$$\langle t_l^{\pm} | \widetilde{A}_{\mathrm{R+P}}(J, t) | t_r \rangle = \frac{F(t_l^{\pm}, t) F(t_r^{\pm}, t) \left[\frac{(J - \alpha_{c_l})(J - \alpha_{c_r})(\alpha + 1)}{(\alpha - \alpha_{c_l})(\alpha - \alpha_{c_r})(J + 1)} \right]}{J - \alpha - \frac{\alpha + 1}{J + 1} \left[(J - \alpha) \int \mathrm{d}\phi^{\pm} \frac{F^2(t^{\pm}, t)}{(\alpha - \alpha_c)^2} + \int \mathrm{d}\phi^{\pm} \frac{F^2(t^{\pm}, t)}{\alpha - \alpha_c} \right]} . \tag{5.2}$$

In the loop integrals there is no phase factor $\cos[\pi(\alpha(t^+) - \alpha(t^-))]$ because the loops have only twisted reggeons [9]. The pole position is determined by

$$0 = J_{\mathrm{P}} - \alpha - \frac{\alpha + 1}{J_{\mathrm{P}} + 1} \left[(J_{\mathrm{P}} - \alpha) \int \mathrm{d}\phi^{\pm} \frac{F^2(t^{\pm}, t)}{(\alpha - \alpha_c)^2} + \int \mathrm{d}\phi^{\pm} \frac{F^2(t^{\pm}, t)}{\alpha - \alpha_c} \right] . \tag{5.3}$$

Let us study this equation at $t = 0$. Assuming the validity of the Rosenzweig–Veneziano condition (eq. (2.11)) and linear trajectories (i.e. $\alpha(t) = \alpha(0) + \alpha' t$) we find

$$[J_{\mathrm{P}} - \alpha(0)]^2 = 1 - \alpha(0)^2 + \int \mathrm{d}\phi^{\pm} \frac{F^2(t^{\pm}, 0)}{(\alpha(0) - \alpha_c)^2} [-2\alpha' t^+] , \tag{5.4}$$

where, of course, $\mathrm{d}\phi^{\pm} = \mathrm{d}t^+ \mathrm{d}t^- \delta(t^+ - t^-)$. The remaining integral gives a measure of the average value of the loop momentum transfer, since without the factor $(-2\alpha' t^+)$ the integral equals unity. Using a reasonable parametrization of the triple-reggeon coupling [9], the integral is estimated to be $\simeq 0.5$; consequently $J_{\mathrm{P}} \simeq 1.62$ which is a rather high bare pomeron intercept. In fact, by neglecting the contribution of the integral in (5.4), we find a lower bound $J_{\mathrm{P}} \geqslant 1.37$. Note, however, that the amplitude (5.1) could be modified by a smooth multiplicative factor $h(J, t)$ normalized so that $h(J = \alpha, t) = 1$. If this function drops by a factor of ~ 2 when J goes from α to ~ 1 then a reasonable pomeron intercept may be obtained.

The one-dimensional limit of (5.2) does *not* correspond to the usual diagram counting form [11]. Identifying $F(t^\pm, 0)$ with the coupling g and using (3.3) gives

$$\langle t_l^\pm | \tilde{A}_{R+P}(J, 0) | t_r^\pm \rangle \xrightarrow{1\,\mathrm{dim}} \frac{g^2 [(J-\alpha_c)^2/(\alpha-\alpha_c)^2](\alpha+1)}{(J-\alpha)^2+\alpha^2-1} . \tag{5.5}$$

The lower bound $J_P = 1.37$ is now attained.

Our second example involves the 'symmetric sum' choice for reggeon-reggeon amplitude [21]

$$\langle t_l^\pm | \tilde{A}(J, t) | t_r^\pm \rangle = \frac{F(t_l^\pm, t) F(t_t^\pm, t)}{J-\alpha} \frac{1}{2}\left[\frac{J-\alpha_{c_l}}{\alpha-\alpha_{c_l}} + \frac{J-\alpha_{c_r}}{\alpha-\alpha_{c_r}}\right] . \tag{5.6}$$

Calculation of the pomeron (again with four external reggeon legs) is rather tedious but straightforward, with the result:

$$\begin{aligned}
&\langle t_l^\pm | \tilde{A}_{R+P}(J, t) | t_r^\pm \rangle \\
&= \frac{F(t_l^\pm, t) F(t_r^\pm, t)}{4} \left\{ 2\left[\frac{J-\alpha_{c_l}}{\alpha-\alpha_{c_l}} + \frac{J-\alpha_{c_r}}{\alpha-\alpha_{c_r}}\right] + Z_2(\alpha, t) + \frac{(J-\alpha) Z_1(\alpha, t)}{(\alpha-\alpha_{c_l})(\alpha-\alpha_{c_r})} \right. \\
&\left. - \left[\frac{J-\alpha_{c_l}}{\alpha-\alpha_{c_l}}\right]\left[\frac{J-\alpha_{c_r}}{\alpha-\alpha_{c_r}}\right] Z_2(J, t) \right\} / \{J - \alpha - Z_1(\alpha, t) - \tfrac{1}{4} Z_1(J, t) Z_2(\alpha, t) \\
&+ \tfrac{1}{4} Z_1(\alpha, t) Z_2(J, t)\} ,
\end{aligned} \tag{5.7}$$

where

$$Z_1(J, t) = \int \mathrm{d}\phi^\pm \frac{F^2(t^\pm, t)}{J-\alpha_c} , \qquad Z_2(J, t) = \int \mathrm{d}\phi^\pm \frac{F^2(t^\pm, t)}{(J-\alpha_c)(\alpha-\alpha_c)} . \tag{5.8}$$

The one-dimensional limit of (5.7) is

$$\langle t_l^\pm | \tilde{A}_{R+P}(J, 0) | t_r^\pm \rangle \xrightarrow{1\,\mathrm{dim}} \frac{g^2 (J-\alpha_c)/(\alpha-\alpha_c)}{J-1} = \frac{2g^2}{J-1} + \frac{g^2}{\alpha-\alpha_c} , \tag{5.9}$$

which indeed has the pomeron with pole position $J_P = 1$ and residue $2g^2$ as found in the simple Huan–Lee model [11].

In the three-dimensional case, the pole position is determined by

$$J_P - \alpha - Z_1(\alpha, t) - \tfrac{1}{4} Z_1(J_P, t) Z_2(\alpha, t) + \tfrac{1}{4} Z_1(\alpha, t) Z_2(J_P, t) = 0 . \tag{5.10}$$

We have studied this equation with the following parametrization for the triple-reggeon coupling [9]:

$$F(t_1^{\pm}, t) = (\alpha - \alpha_{c_1}) f e^{at/2} e^{b(t_1^+ + t_1^-)/2} , \qquad (5.11)$$

where *a, b, f* are interrelated through the planar bootstrap. The pomeron intercept $\alpha_P(0)$ and slope $\alpha'_P(0)$ are very sensitive to the values of these parameters. Reasonable values for $\alpha_P(0)$ and $\alpha'_P(0)$ can be obtained for $b/\alpha' \simeq 3$. This is somewhat higher than $b/\alpha' \simeq 2$ suggested in Bishari's calculation [9] using a reggeon-reggeon amplitude having only a pure pole and no other J dependence.

Our examples illustrate how sensitively the pomeron depends on the details of the reggeon-reggeon amplitude. These details include not only the parametrization of the triple-reggeon couplings, but, more important, the smooth J dependence in $\langle t_l^{\pm} | \tilde{A}(J, t) | t_r^{\pm} \rangle$ in addition to the pure pole. This smooth J dependence cannot be ignored as it is usually crucial to the bootstrap of the planar amplitude.

6. Remarks and conclusions

In this paper we have shown how very important the choice of NDC and precise analytic behaviour of input production amplitudes are for the dual bootstrap scheme – even in the one-dimensional cases. Consistency requires not only pole position and residue matching but also the absence of output cuts. Schemes that fail to kill the cuts may still provide consistency at the pole but usually at the expense of $\bar{L}$ dependent conditions. But, to the extent that the cut-off $\bar{L}$ is viewed as a mere counting device, the pole position should be determined by an *$\bar{L}$-independent* condition. This view point is natural in the approach where clusters are chosen as groups of particles with overall rapidity size up to $\bar{L}$. In this case the NDC are determined by the manner in which grouping of particles in the unitarity sum is accomplished. In particular, the construction of clusters shown in fig. 1 and described in sect. 2 corresponds to the asymmetric NDC of sect. 4. If on the other hand, clusters are viewed as narrow resonances or $q\bar{q}$ pairs, then $\bar{L}$ may have a physical significance since it shows up in the cluster multiplicity [16]. For such dynamical pictures of the intermediate states in hadronic production, the symmetric NDC of appendix C is perhaps a natural choice.

Let us summarize some relevant conclusions and comments:

(i) Care is needed in dealing with singular amplitudes. Keeping track of the singular nature of amplitudes is also necessary in a simple generalization of the original Rosenzweig–Veneziano scheme [5] when phase space is divided into three or more integration regions. Work in progress shows that provided the analytic structure of dual amplitudes is carefully respected, no difficulty arises in achieving consistency for the planar bootstrap [26]. On the other hand, if explicit, overly simple, forms for the cluster amplitudes are assumed difficulties inevitably arise and only approximate consistency may be achieved [19,22,24].

(ii) The simple NDC discussed in appendix A (minimal gap size, maximal cluster size) guarantees strict no double counting at the expense of leaving out configurations with small gaps. Complete consistency (cut-killing included) seems to be impossible.

(iii) The asymmetric NDC of sect. 4 leads to the Rosenzweig–Veneziano condition [5] and cut-cancellation independent of the maximum cluster size (for reggeon-reggeon amplitudes as well). The asymmetry is a mere reflection of proper counting of events.

(iv) Studying the asymmetric NDC bootstrap in terms of rapidities rather than in the j plane reveals interesting points whose details will be described elsewhere [26]. (i) Due to the singular nature of amplitudes at threshold, care should be taken in integrations over cluster lengths and gaps. The natural order of integration over clusters and gaps (which is the reverse of the order of assignment of clusters) gives the self-consistent result. If a different order of integration is chosen one may easily lose or double count some of these singular contributions. (ii) At any given $Y \geqslant \bar{L}$ summation over all terms contributing at that value of Y (i.e. number of clusters $\leqslant [Y/\bar{L}] + 1$) shows that the amplitude minus its Born term has a simple Regge behaviour ($e^{Y\alpha}$) provided the Rosenzweig–Veneziano condition [5] is satisfied. Obviously each term in the sum has contributions (i.e. cut-terms) deviating from this behaviour; but they are all cancelled in the sum (even though it is a finite sum). Thus the order of summation is important.

(v) Using the symmetric NDC of appendix C, we found that the unitarity equations easily accommodate factorization of the output amplitudes. However, the bootstrap condition is $\bar{L}$ dependent and cut-killing (if at all possible) is complicated for $\bar{L} \neq 0$. At $\bar{L} = 0$ self-consistency is achieved.

(vi) Testing consistency in $\bar{L} = 0$ models may be misleading: it does not necessairly imply consistency for $\bar{L} \neq 0$.

(vii) Our approach has limitations of kinematic nature in that multi-Regge phase space is used. Thus, $t_{\min}$ effects are neglected. However, the NDC imply repulsion between clusters and may partially compensate for $t_{\min}$ effects. We use the "hybrid" set of variables: rapidity and momentum transfer. However rapidities are easily translatable into sub-energies in the multi-Regge limit. Finally, we assume single reggeon exchange even across small gaps.

(viii) At least in dual theories the planar amplitude is given, even for small gaps, by a sum over a countable number of poles. We have studied the effect of introducing any countable number of poles on the bootstrap scheme. In general, the bootstrap equations become matrix equations. With the asymmetric NDC we find the same results as in the single pole case of sect. 4, except for additional orthogonality conditions similar to those obtained by Bishari [25]. These orthogonality constraints affect triple-reggeon couplings involving reggeons of different intercepts.

(ix) The "conventional" bare pomeron is very sensitive to the precise analytic structure of the planar amplitudes. Slight alterations introduced by changing the smooth J-dependent terms of the planar amplitude (these terms are usually necessary in addition to the pole term) have a significant effect on the pomeron parameters. However, the calculational procedure of the bare pomeron [2,7,9] used also by us in sect. 5 does not strictly respect FMSR for the planar amplitudes. The latter play a crucial role in the planar bootstrap.

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Appendix A

Maximum cluster size and minimum gap size

We consider here a condition on the cluster phase space that does not explicitly correlate clusters and gaps, namely

$$\theta(\bar{L} - L_i)\,\theta(g_i - \bar{g})\,.$$

By taking $\bar{L} = \bar{g}$, this condition strictly avoids double-counting but does so at the expense of leaving out part of the possible phase space configurations (i.e. gaps whose size is less than $\bar{L}$). The unitarity equation for the particle-particle amplitude is

$$\langle b|\tilde{A}(J,t)|a\rangle - \langle b|\tilde{A}^{\bar{L}}(J,t)|a\rangle = \langle b|\tilde{T}_2(J,t,\bar{L})|a\rangle$$

$$+\int \mathrm{d}\phi_1^{\pm}\langle b|\tilde{G}^{\bar{L}}(J,t)|t_1^{\pm}\rangle\,\tilde{X}(J,t_1^{\pm})\,\mathrm{e}^{-\bar{L}(J-\alpha_{\mathrm{c}_1})}\langle t_1^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle\,. \qquad \text{(A.1)}$$

Similar equations hold for reggeon-particle (reggeon-reggeon) amplitudes. $\langle b|\tilde{T}_2(J,t,\bar{L})|a\rangle$ is the two cluster contribution and $\langle t_l^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle$ includes chains with at least two clusters. The two-cluster term is explicitly separated since, in general, when amplitudes are singular at threshold (e.g., if FMSR (4.6) are assumed) it cannot be written in the standard factorized form:

$$\langle b|\tilde{T}_2(J,t,\bar{L})|a\rangle \neq \int \mathrm{d}\phi_1^{\pm}\langle b|\tilde{G}^{\bar{L}}(J,t)|t_1^{\pm}\rangle\,\tilde{X}(J,t_1^{\pm})\,\mathrm{e}^{-\bar{L}(J-\alpha_{\mathrm{c}_1})}\langle t_1^{\pm}|\tilde{G}^{\bar{L}}(J,t)|a\rangle\,.$$

However, its somewhat complicated form can be calculated using FMSR (4.6). The recursion relation satisfied by $\langle t_l^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle$ is

$$\langle t_l^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle = \langle t_l^{\pm}|\tilde{T}_2(J,t,\bar{L})|a\rangle$$

$$+\int \mathrm{d}\phi_1^{\pm}\langle t_l^{\pm}|\tilde{G}^{\bar{L}}(J,t)|t_1^{\pm}\rangle\,\tilde{X}(J,t_1^{\pm})\,\mathrm{e}^{-\bar{L}(J-\alpha_{\mathrm{c}_1})}\langle t_1^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle\,. \qquad \text{(A.3)}$$

Eq. (A.3) implies when compared to eq. (A.1) (with b replaced by a reggeon leg):

$$\langle t_l^{\pm}|\tilde{T}(J,t,\bar{L})|a\rangle = \langle t_l^{\pm}|\tilde{A}(J,t)|a\rangle - \langle t_l^{\pm}|A^{\bar{L}}(J,t)|a\rangle\,. \qquad \text{(A.4)}$$

Thus, (A.1) is not an integral equation for amplitudes but for amplitudes minus their Born terms. Demanding that the output amplitudes be simple poles (for $Y \geqslant \bar{L}$) we impose eq. (2.10) on the r.h.s. of eq. (A.4) and find

$$\langle t_l^\pm | \widetilde{T}(J, t, \bar{L}) | a \rangle = \frac{F(t_l^\pm, t) F_a(t)}{J - \alpha} \, e^{-\bar{L}(J-\alpha)} \, . \tag{A.5}$$

Inserting (A.5) in (A.3) we find

$$\frac{F(t_l^\pm, t) F_a(t)}{J - \alpha} e^{-\bar{L}(J-\alpha)} = \frac{\langle t_l^\pm | \widetilde{T}_2(J, t, \bar{L}) | a \rangle}{1 - H(J, t, \bar{L}; t_l^\pm, a)} \, , \tag{A.6}$$

where

$$H(J, t, \bar{L}; t_l^\pm, a) = \int d\phi_1^\pm \frac{\langle t_l^\pm | \widetilde{G}^{\bar{L}}(J, t) | t_1^\pm \rangle}{F(t_l^\pm, t)} \widetilde{X}(J, t_1^\pm) \, e^{-\bar{L}(J-\alpha_{c_1})} F(t_1^\pm, t)$$

$$= \int d\phi_1^\pm F(t_1^\pm, t) \, e^{-\bar{L}(J-\alpha_{c_1})} \widetilde{X}(J, t_1^\pm) \frac{\langle t_1^\pm | \widetilde{G}^{\bar{L}}(J, t) | a \rangle}{F_a(t)} \, . \tag{A.7}$$

For reggeon-reggeon amplitudes a is replaced in eqs. (A.6–A.7) by a reggeon leg ($t_r^\pm$). The pole position constraint is:

$$H(\alpha, t, \bar{L}; t_l^\pm, a) = H(\alpha, t, \bar{L}; b, t_r^\pm) = H(\alpha, t, L; t_l^\pm, t_r^\pm) = 1 \, . \tag{A.8}$$

This constraint should be independent of the external legs, but may in general, depend on $\bar{L}$. The residue matching condition is complicated and intractable.

To obtain the Rosenzweig–Veneziano condition [5] (eq. (2.11)) the integrand in (A.7) must have a peculiar asymmetric form at $J = \alpha$. For instance, one of the forms

$$\langle t_1^\pm | \widetilde{G}^{\bar{L}}(J = \alpha, t) | t_2^\pm \rangle = F(t_1^\pm, t) F(t_2^\pm, t) \begin{cases} \dfrac{e^{(\alpha - \alpha_{c_1})\bar{L}}}{\alpha - \alpha_{c_1}} \\ \\ \dfrac{e^{(\alpha - \alpha_{c_2})\bar{L}}}{\alpha - \alpha_{c_2}} \end{cases} \, , \tag{A.9}$$

would yield eq. (2.11). However, $\langle t_1^\pm | \widetilde{G}^{\bar{L}}(J, t) | t_2 \rangle$ is the cut-off Mellin transform over a reggeon-reggeon amplitudes with free legs on both sides. As such it cannot be asymmetric. Note that the $\bar{L} = 0$ limit of this *ad hoc* form is the t-dependent analogue of the one dimensional model [11] discussed in sect. 3.

If one abandons FMSR (4.6) and assumes that the input cluster amplitudes are smooth function of L, eqs. (A.1–6) can be recast in the standard factorized Chew–Goldberger–Low form [10]. For a simple choice:

$$\langle t_1^\pm | \widetilde{G}^{\bar{L}}(J, t) | t_2^\pm \rangle = F(t_1^\pm, t) \, F(t_2^\pm, t) \, \widetilde{V}^{\bar{L}}(J, t) \, , \tag{A.10}$$

(i.e., external leg dependence only in couplings) one can explicitly show that cut killing is impossible.

Appendix B

Problems with reggeon-reggeon amplitudes

Much case is required in dealing with Mellin transforms over reggeon-reggeon amplitudes [20,21]. Such amplitudes actually depend on many independent variables and various Mellin transforms (with related FMSR) over the missing mass variable (or the L variable) can be defined, depending on which of the other variables are held fixed [20,21].

In this appendix we show why caution is needed in manipulations involving a cluster that has only reggeon legs and that seems "unconstrained" by NDC (4.1). Suppose the reggeon-particle amplitude is built up following fig. 5b, leaving the end cluster with external reggeons unconstrained by the NDC. If one ignores the singular structure of cluster amplitudes at $L = 0$, eq. (4.12) reads

$$\langle t_1^\pm|\widetilde{A}(J,t)|a\rangle - \langle t^\pm|\widetilde{A}^{\overline{L}}(J,t)|a\rangle$$

$$= \frac{F_a(t)\,e^{-\overline{L}(J-\alpha)}\int d\phi_2^\pm\langle t_1|\widetilde{G}^{\overline{L}}(J,t)|t_2^\pm\rangle F(t_2^\pm,t)\,\widetilde{X}(J,t_2^\pm)/(\alpha-\alpha_{c_2})}{1-e^{-\overline{L}(J-\alpha)}I(J,t)}\,, \qquad \text{(B.1)}$$

where $\langle t_1^\pm|\widetilde{B}^{\overline{L}}(J,t)|t_2^\pm\rangle$ of eq. (4.12) is not distinguished from the cut-off Mellin transform over a cluster with free external reggeon legs $\langle t_1^\pm|\widetilde{G}^{\overline{L}}(J,t)|t_2^\pm\rangle$.

For the l.h.s. of (B.1) one may substitute eq. (2.10). When eq. (B.1) is continued in J to α_{c_1}, $\langle t_1^\pm|\widetilde{G}^{\overline{L}}(J,t)|t_2^\pm\rangle$ continues into the FMSR (4.6b), then

$$-F(t_1^\pm,t)\,F_a(t)\,\frac{e^{\overline{L}(\alpha-\alpha_{c_1})}}{\alpha-\alpha_{c_1}} = \left[F(t_1^\pm,t)\,F_a(t)\,\frac{e^{\overline{L}(\alpha-\alpha_{c_1})}}{\alpha-\alpha_{c_1}}\right]$$

$$\times\frac{e^{\overline{L}(\alpha-\alpha_{c_1})}I(J=\alpha_{c_1},t)}{1-e^{\overline{L}(\alpha-\alpha_{c_1})}I(J=\alpha_{c_1},t)}\,. \qquad \text{(B.2)}$$

Unless $I(J=\alpha_{c_1},t)$ has a very pathological behaviour, eq. (B.2) leads to a contradiction. Furthermore, the simplest symmetric candidates fail to solve (B.1). These are the symmetric product [9]:

$$\langle t_1^\pm|\widetilde{G}^{\overline{L}}(J,t)|t_2^\pm\rangle = \frac{F(t_1^\pm,t)\,F(t_2^\pm,t)}{J-\alpha}\,[h(J,t)-e^{-\overline{L}(J-\alpha)}]\,, \qquad \text{(B.3)}$$

($h(J, t)$ is a smooth function satisfying $h(J = \alpha, t) = 1$); the symmetric product:

$$\langle t_1^{\pm} | \widetilde{G}^{\overline{L}}(J, t) | t_2^{\pm} \rangle = \frac{F(t_1^{\pm}, t) F(t_2^{\pm}, t)}{J - \alpha} \left[\frac{J - \alpha_{c_1}}{\alpha - \alpha_{c_1}} \frac{J - \alpha_{c_2}}{\alpha - \alpha_{c_2}} h(J, t) - e^{-\overline{L}(J-\alpha)} \right], \tag{B.4}$$

the symmetric sum [21]

$$\langle t_1^{\pm} | \widetilde{G}^{\overline{L}}(J, t) | t_2^{\pm} \rangle = \frac{F(t_1^{\pm}, t) F(t_2^{\pm}, t)}{J - \alpha} \left[\left\{ \frac{J - \alpha_{c_1}}{\alpha - \alpha_{c_1}} + \frac{J - \alpha_{c_2}}{\alpha - \alpha_{c_2}} \right\} h(J, t) - e^{-\overline{L}(J-\alpha)} \right].$$

However the combination

$$\langle t_1^{\pm} | \widetilde{G}^{\overline{L}}(J, t) | t_2^{\pm} \rangle = \frac{F(t_1^{\pm}, t) F(t_2^{\pm}, t)}{J - \alpha} \left\{ h_1(J, t) \left[\frac{J - \alpha_{c_1}}{\alpha - \alpha_{c_1}} + \frac{J - \alpha_{c_2}}{\alpha - \alpha_{c_2}} \right] \right.$$

$$\left. + h_2(J, t) \left[\frac{J - \alpha_{c_1}}{\alpha - \alpha_{c_1}} \frac{J - \alpha_{c_2}}{\alpha - \alpha_{c_2}} \right] + h_3(J, t) - e^{-\overline{L}(J-\alpha)} \right\}, \tag{B.6}$$

is a possible solution of eq. (B.1) provided the following relations hold:

$$h_1(J, t) = 1 - h_3(J, t) I(J, t), \qquad h_2(J, t) = -I(J, t) + h_3(J, t) I^2(J, t). \tag{B.7}$$

This of course introduces cuts (through $I(J, t)$) in eq. (B.6) and is therefore not an acceptable solution (interestingly, for an appropriate choice of $h_3(J, t)$ the reggeon-reggeon "amplitude" corresponding to eq. (B.6) continues smoothly to the cut-free reggeon-particle and particle-particle amplitudes of ref. [8]).

These difficulties arise when the cut-off Mellin transform of a reggeon-reggeon amplitude whose legs on both sides are free is identified with $\langle t_1^{\pm} | \widetilde{B}^{\overline{L}}(J, t) | t_2^{\pm} \rangle$. The latter is a particular integral over a cluster, which is not directly affected by the NDC, but is affected by the NDC applied to all other clusters and gaps, and by the overall energy conservation constraint.

The foregoing illustrates why caution is needed when handling Mellin transforms (cut-off or not) over reggeon-reggeon amplitudes. The 'naive' identification of $\langle t_1^{\pm} | \widetilde{B}^{\overline{L}}(J, t) | t_2^{\pm} \rangle$ with $\langle t_1^{\pm} | \widetilde{G}^{\overline{L}}(J, t) | t_2^{\pm} \rangle$ ignores certain subtleties related to the analytic structure of amplitudes. One major subtle point overlooked in such a case is the problem of guaranteeing strict no double counting (even for zero measure integrations). Substituting $\langle t_1^{\pm} | \widetilde{G}^{\overline{L}}(J, t) | t_2^{\pm} \rangle$ for $\langle t_1^{\pm} | \widetilde{B}^{\overline{L}}(J, t) | t_2 \rangle$ in eq. (B.1) one includes a disconnected part (corresponding to a situation in which the "left over" cluster is empty; this never occurs with clusters having external particle legs). This mistake follows from glossing over the delicacy of integrations over cluster amplitudes which are singular at threshold.

Note that in the calculational procedure of sect. 4 these subtleties have been properly handled (by carefully keeping track of limits of integration over cluster amplitudes and reggeon exchanges across gaps).

In light of this discussion it is not surprising that the "solution" (B.6) has cuts.

Appendix C

Gap-cluster correlations via symmetric NDC

We consider here the symmetric NDC

$$L_{i-1}+g_i+L_i \geqslant \bar{L} \ . \tag{C.1}$$

Undoing a few integrations from one edge of the chain shown in fig. 2, we write the following integral expression for the particle-particle amplitude:

$$\begin{aligned}\langle b|\tilde{A}(J,t)|a\rangle - \langle b|\tilde{A}^{\bar{L}}(J,t)|a\rangle &= \langle b|\tilde{T}_2(J,t)|a\rangle \\ &+\int \mathrm{d}\phi_1^{\pm} \int_0^{\bar{L}} \mathrm{d}L_0\, \mathrm{e}^{-JL_0} \langle b|G(L_0,t)|t_1^{\pm}\rangle \int_0^{\infty} \mathrm{d}g_1\, \mathrm{e}^{-g_1(J+1)} X(g_1,t_1^{\pm}) \\ &\times \int_0^{\bar{L}} \mathrm{d}L_1\, \mathrm{e}^{-JL_1} \langle L_1 t_1^{\pm}|T(J,t)|a\rangle\, \theta(L_0+g_1+L_1-\bar{L}) \ .\end{aligned} \tag{C.2}$$

Here $\langle L_1 t_1^{\pm}|T(J,t)|a\rangle$ is a sum over chains ending with reggeon legs on the left and with the L integration over the first cluster along each chain undone. Its recursion relation is given further on.

Here again (as in appendix A) the two-cluster term has to be separated out. Explicit calculation (using FMSR (4.6)) shows that all the dependence of this term on the external legs factorizes out. Otherwise, it is not of a simple form. It has cuts, but is regular at $J=\alpha$. The $\bar{L}$ dependence in $\tilde{T}_2$ and T of eq. (C.2) is suppressed.

Performing the gap integration in eq. (C.2) we find

$$\begin{aligned}\langle b|\tilde{A}(J,t)|a\rangle - \langle b|\tilde{A}^{\bar{L}}(J,t)|a\rangle &= \langle b|\tilde{T}_2(J,t)|a\rangle \\ &+\int \mathrm{d}\phi_1^{\pm}\, \tilde{X}(J,t_1^{\pm}) \int_0^{\bar{L}} \mathrm{d}L_1 \mathrm{e}^{-L_1 J} \left\{ \int_{\bar{L}-L_1}^{\bar{L}} \mathrm{d}L_0 \mathrm{e}^{-L_0 J} \langle b|G(L_0,t)|t_1^{\pm}\rangle \right. \\ &\left. + \mathrm{e}^{-(\bar{L}-L_1)(J-\alpha_{c_1})} \int_0^{\bar{L}-L_1} \mathrm{d}L_0\, \mathrm{e}^{-L_0\alpha_{c_1}} \langle b|G(L_0,t)|t_1^{\pm}\rangle \right\} \\ &\times \langle L_1 t_1^{\pm}|T(J,t)|a\rangle \ .\end{aligned} \tag{C.3}$$

The first integral in the curly brackets is just the difference of two cut-off Mellin transforms over a reggeon-particle amplitude; the second integral is just a FMSR for which we assume the form given in eq. (4.6a). Thus eq. (C.3) becomes

$$\langle b|\widetilde{A}(J,t)|a\rangle - \langle b|\widetilde{A}^{\bar{L}}(J,t)|a\rangle = \langle b|\widetilde{T}_2(J,t)|a\rangle$$

$$+\int \mathrm{d}\phi_1^\pm \widetilde{X}(J,t_1^\pm)\int_0^{\bar{L}} \mathrm{d}L_1 \mathrm{e}^{-L_1 J} F_\mathrm{b}(t) F(t_1^\pm, t)\left\{\frac{-\mathrm{e}^{-\bar{L}(J-\alpha)} + \mathrm{e}^{-(\bar{L}-L_1)(J-\alpha)}}{J-\alpha}\right.$$

$$\left. + \frac{\mathrm{e}^{-(\bar{L}-L_1)(J-\alpha_{\mathrm{c}_1})}\,\mathrm{e}^{(\bar{L}-L_1)(\alpha-\alpha_{\mathrm{c}_1})}}{\alpha-\alpha_{\mathrm{c}_1}}\right\}\langle L_1 t_1^\pm|T(J,t)|a\rangle\,. \tag{C.4}$$

We may write this as follows:

$$\langle b|\widetilde{A}(J,t)|a\rangle - \langle b|\widetilde{A}^{\bar{L}}(J,t)|a\rangle = \langle b|\widetilde{T}_2(J,t)|a\rangle + F_\mathrm{b}(t)\,\frac{\mathrm{e}^{-\bar{L}(J-\alpha)}}{J-\alpha}\,Z(J,\bar{L},t,a)\,, \tag{C.5}$$

where

$$Z(J,\bar{L},t,a) \equiv \int \mathrm{d}\phi_1^\pm F(t_1^\pm,t)\,\widetilde{X}(J,t_1^\pm)\left\{\frac{J-\alpha_{\mathrm{c}_1}}{\alpha-\alpha_{\mathrm{c}_1}}\int_0^{\bar{L}} \mathrm{d}L_1 \mathrm{e}^{-\alpha L_1}\langle L_1 t_1^\pm|T(J,t)|a\rangle\right.$$

$$\left. - \int_0^{\bar{L}} \mathrm{d}L_1\,\mathrm{e}^{-JL_1}\langle L_1 t_1^\pm|T(J,t)|a\rangle\right\}\,. \tag{C.6}$$

A similar relation can be written with the roles of b and a interchanged. Now the recursion relation satisfied by $\langle L_1 t_1^\pm|T(J,t)|a\rangle$ is

$$\langle L_1 t_1^\pm|T(J,t)|a\rangle - \langle L_1 t_1^\pm|T_2(J,t)|a\rangle$$

$$= \int \mathrm{d}\phi_2^\pm\,\widetilde{X}(J,t_2^\pm)\int_0^{\bar{L}} \mathrm{d}L_2 \mathrm{e}^{-L_2 J}\{\theta(L_1+L_2-\bar{L}) + \theta(\bar{L}-L_1-L_2)$$

$$\times\,\mathrm{e}^{-(\bar{L}-L_1-L_2)(J-\alpha_{\mathrm{c}_1})}\}\langle t_1^\pm|G(L_1,t)|t_2^\pm\rangle\langle L_2 t_2^\pm|T(J,t)|a\rangle\,, \tag{C.7}$$

where $\langle L_1 t_1^\pm|T_2(J,t)|a\rangle$ satisfies

$$\int_0^{\bar{L}} \mathrm{d}L_1\,\mathrm{e}^{-JL_1}\langle L_1 t_1^\pm|T_2(J,t)|a\rangle = \langle t_1^\pm|\widetilde{T}_2(J,t)|a\rangle\,. \tag{C.8}$$

The reggeon-particle amplitude is constructed as follows: undo in every chain the first cluster and first gap integrations, cancel the first NDC in the chain, then use factorization to get rid of all first cluster and first gap factors. It satisfies equations similar to (C.4-5) with b replaced by an external reggeon. Defining

$$\langle t_1^{\pm}|\widetilde{T}(J,t)|a\rangle = \int_0^{\bar{L}} \mathrm{d}L_1\, \mathrm{e}^{-L_1 J}\langle L_1 t_1^{\pm}|\widetilde{T}(J,t)|a\rangle\,, \tag{C.9}$$

we find

$$\langle t_1^{\pm}|\widetilde{T}(J,t)|a\rangle = \langle t_1^{\pm}|\widetilde{A}(J,t)|a\rangle - \langle t_1^{\pm}|\widetilde{A}^{\bar{L}}(J,t)|a\rangle$$

$$= \langle t_1^{\pm}|\widetilde{T}_2(J,t)|a\rangle + \frac{F(t_1^{\pm},t)\,\mathrm{e}^{-\bar{L}(J-\alpha)}}{J-\alpha}\, Z(J,\bar{L},t,a)\,. \tag{C.10}$$

A similar equation holds for the reggeon-reggeon amplitude. Thus, the unitarity equations demand that (recall that the two-cluster term factorizes in the external legs)

$$\frac{\langle l|\widetilde{A}(J,t)|r\rangle - \langle l|\widetilde{A}^{\bar{L}}(J,t)|r\rangle}{F_l(t)F_r(t)} = w(J,\bar{L},t)\,, \tag{C.11}$$

for l, r being reggeon or particle legs. Thus, all the dependence on external legs is in the couplings, and the output amplitudes minus their Born terms exhibit the desired factorization properties.

We do not solve here the complicated recursion relation (C.7). Instead, let us look at the implicit bootstrap requirement

$$F_{\mathrm{a}}(t) = Z(J,\bar{L},t,a) + (J-\alpha)\,\mathrm{e}^{\bar{L}(J-\alpha)}\langle b|\widetilde{T}_2(J,t)|a\rangle/F_{\mathrm{b}}(t)\,. \tag{C.12}$$

At $J=\alpha$, residue consistency therefore demands

$$F_{\mathrm{a}}(t) = \Big[\int \mathrm{d}\phi_1^{\pm} F(t_1^{\pm},t)\,\widetilde{X}(\alpha,t_1^{\pm})$$
$$\times \lim_{J\to\alpha}\left\{\frac{(J-\alpha)}{\alpha-\alpha_{\mathrm{c}_1}}\int_0^{\bar{L}} \mathrm{d}L_1 \mathrm{e}^{-\alpha L_1} + \int_0^{\bar{L}} \mathrm{d}L_1\,[\mathrm{e}^{-\alpha L_1} - \mathrm{e}^{-JL_1}]\right\}$$
$$\times \langle L_1 t_1^{\pm}|T(J,t)|a\rangle \quad . \tag{C.13}$$

From eq. (C.10) it is obvious that

$$\langle L_1 t_1^{\pm}|T(J,t)|a\rangle = \langle L_1 t_1^{\pm}|V(J,t)|a\rangle/(J-\alpha)\,, \tag{C.14}$$

where $\langle L_1 t_1^{\pm}|V(J,t)|a\rangle$ is a smooth function of J and at $J=\alpha$ is positive definite for $t=0$. It satisfies

$$\int_0^{\bar{L}} \mathrm{d}L_1\,\mathrm{e}^{-\alpha L_1}\langle L_1 t_1^{\pm}|V(\alpha,t)|a\rangle = F(t_1^{\pm},t)\,F_{\mathrm{a}}(t)\,. \tag{C.15}$$

Thus eq. (C.13) becomes

$$1=\int \mathrm{d}\phi_1^{\pm}\frac{F^2(t_1^{\pm},t)\widetilde{X}(\alpha,t_1^{\pm})}{\alpha-\alpha_{c_1}}+\int \mathrm{d}\phi_1^{\pm}F(t_1^{\pm},t)\widetilde{X}(\alpha,t_1^{\pm})\int_0^{\bar{L}}\mathrm{d}L_1\,\mathrm{e}^{-\alpha L_1}$$

$$\times L_1\frac{\langle L_1t_1^{\pm}|V(\alpha,t)|a\rangle}{F_\mathrm{a}(t)}\,. \qquad \text{(C.16)}$$

Thus the Rosenzweig–Veneziano condition [5] results *provided*

$$\int_0^{\bar{L}}\mathrm{d}L_1\mathrm{e}^{-\alpha L_1}L_1\langle L_1t_1^{\pm}|V(\alpha,t)|a\rangle=0\,. \qquad \text{(C.17)}$$

At $t=0$, the integrands of eqs. (C.15) and (C.17) are non-negative. Therefore, eq. (C.17) cannot hold for any $\bar{L}$ except for $\bar{L}=0$.

Furthermore, one can show that the $\bar{L}=0$ limit of the integral equations of the present model can be cast in the form of the $\bar{L}=0$ limit of the equations derived in the model of sect. 4. Hence cut-killing is realized with the symmetric NDC discussed here, for $\bar{L}=0$. Obviously, the one-dimensional $\bar{L}=0$ limit reduces to the standard one-dimensional model [11]. For $\bar{L}\neq 0$ the bootstrap condition equation (C.16) is $\bar{L}$ dependent. This is in accord with our remark that the NDC eq. (C.1) corresponds naturally to dynamical models where $\bar{L}$ has a physical significance and is not a mere bookeeping device; cut-killing with an $\bar{L}$ -dependent bootstrap as is the case here is still an open problem.

NDC (C.1) has been discussed in literature with the FMSR (4.6) replaced by an explicit form for the input cluster amplitudes [19,22]:

$$\langle i|G(L,t)|k\rangle=F(i,t)F(k,t)\mathrm{e}^{L\,\alpha(t)}\,. \qquad \text{(C.18)}$$

This form correctly describes only the high mass (i.e., high L) behaviour of the amplitudes but ignores their low-energy singularities. The solution obtained in the one dimensional case [18,22] requires an $\bar{L}$-dependent bootstrap. Cut killing is impossible. Unlike the earlier example (eqs. (C.2) to (C.17)), where FMSR were respected, here there is no possibility for self-consistency even for $\bar{L}=0$ (the output amplitude vanishes!).

In conclusion, we see that the input analyticity assumption plays a crucial role. Moreover, consistency for $\bar{L}=0$ (even when analyticity is respected) does not guarantee consistency for $\bar{L}\neq 0$.

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