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Reggeon amplitudes satisfying good finite-mass sum rules*

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We develop a set of sufficiency conditions which guarantee that certain Reggeon amplitudes satisfy good finite-mass sum rules. We present and analyze a specific and simple model amplitude which embodies these conditions.

I. INTRODUCTION

In this paper we address ourselves to the task of building a model for a Reggeon + Reggeon → particle + particle planar amplitude that satisfies a good finite-mass sum rule (FMSR). As has been discussed in the preceding paper[1] (FJ), certain good FMSR's are vital for planar Reggeon amplitudes that enter into planar unitarity if a self-consistent planar pole bootstrap is to be achieved. Although, such a planar bootstrap is the foundation on which the topological expansion[2] (TE) rests, up to now there have been no adequate candidates for planar amplitudes that meet all the self-consistency requirements.

Since particular "good" FMSR's are a necessary ingredient of planar amplitudes if Regge cuts are to be canceled in the planar bootstrap, we may use this property as a filter for candidates for the planar S matrix of TE. In addition to good FMSR's, self-consistent planar amplitudes should satisfy the demands of Regge asymptotic behavior and general analytic requirements such as the Steinmann relations. Whether all these requirements are in fact mutually compatible is a nontrivial issue. For example in FJ, we have shown that a rather unconventional asymptotic behavior is a prerequisite for good FMSR's in certain Reggeon amplitudes. This criterion rules out the dual resonance model as a viable candidate for self-consistent planar amplitudes.

We begin, in Sec. II, by considering the general structure of the planar six-point function of Fig. 1. We present a set of sufficient conditions on the amplitude that ensures a good FMSR. In Sec. III, we present and analyze a simple model for the Reggeon + Reggeon → particle + particle planar amplitude that has the necessary analyticity properties and that satisfies a good FMSR. In Sec. IV our simple model for A_6 is briefly compared with the dual resonance model.

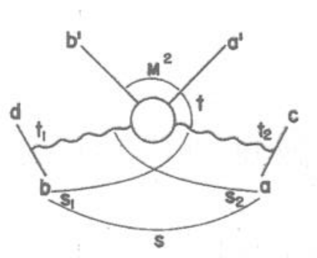

FIG. 1. Six-point function in the limit s_1, s_2, $s \to \infty$ with $s_{a'c}$, $s_{db'}$ fixed.

II. SUFFICIENT CONDITIONS FOR GOOD FMSR'S IN A SIX-POINT FUNCTION

We begin by formulating the general structure of A_6 of Fig. 1. We first lump together particles b' and a' into a single "particle" of mass M^2 and then consider the general Regge formula for the resulting five-point function. (Since A_6 includes only those pieces of the six-point function having physical M^2 singularities, no generality is lost in lumping b' and a' together.) Weis has developed a general representation for such a five-point function incorporating Regge behavior, Steinmann conditions, and the conditions for correct particle pole residues.[3] Adapting this representation to Fig. 1 we write

$$A_6=\beta(b,d;t)\beta(a,c;t_2)\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))\times(-s_1)^{\alpha_1(t_1)}(-s_2)^{\alpha_2(t_2)}\,V\,,$$

$$V=\int_{-i\infty}^{+i\infty}\frac{d\lambda}{2\pi i}\,\Gamma(-\lambda)\frac{\Gamma(\lambda-\alpha_1(t_1))\Gamma(\lambda-\alpha_2(t_2))}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))}\times(-\kappa)^{-\lambda}Q(-M^2;-\lambda;t,t_1,t_2)\,, \qquad (2.1)$$

where $\kappa=s_1s_2/s=M^2+\epsilon$ and ϵ is a non-negative function of the fixed momentum-transfer variables and where the β's are Regge residues. The function Q contains only the physical M^2 cut. The general form (2.1) encompasses a variety of models depending on the detailed dependence of Q on M^2 and on the complex helicity λ. For example, in the case of the dual resonance model

$$Q=Q_{\rm DRM}=\gamma^2B(-\alpha(t)+\alpha_1(t_1)+\alpha_2(t_2)-\lambda\,,\,-\alpha(M^2))\,,$$

where γ is the dual coupling constant. For the present let us maintain as general a form for Q

as possible consistent with general analytic requirements.

To ensure a Steinmann decomposition for V—that is,

$$V=(-\kappa)^{-\alpha_1(t_1)}V_{12}(M^2,\kappa;t,t_1,t_2)+(-\kappa)^{-\alpha_2(t_2)}V_{21}(M^2,\kappa;t,t_1,t_2)\,, \qquad (2.2)$$

where V_{12}, V_{21} are analytic in κ—Q must not have any λ singularities for $\lambda<0$ and the integrand of (2.1) must converge rapidly enough to allow the contour to be closed to the left. With this assumption we pick up the poles at $\lambda=\alpha_1(t_1)-n$, where $n=0,+1,+2,\ldots$, and may express the Steinmann piece corresponding to simultaneous cuts in s and s_2 in the amplitude A_6 as

$$V_{12}=\frac{\pi}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))\sin\pi(+\alpha_1(t_1)-\alpha_2(t_2))}\sum_{n=0}^{\infty}\frac{\Gamma(n-\alpha_1(t_1))(-\kappa)^nQ(-M^2;n-\alpha_1(t_1);t,t_1,t_2)}{\Gamma(n+1)\Gamma(n+1+\alpha_2(t_2)-\alpha_1(t_1))}\,. \qquad (2.3)$$

Alternatively it will prove convenient to employ the identity

$$1=\frac{e^{i\pi(\alpha_1(t_1)-\lambda)}\sin\pi(\lambda-\alpha_2(t_2))-e^{i\pi(\alpha_2(t_2)-\lambda)}\sin\pi(\lambda-\alpha_1(t_1))}{\sin\pi(-\alpha_2(t_2)+\alpha_1(t_1))}$$

to write

$$V_{12}=\frac{(\kappa)^{\alpha_1(t_1)}}{\sin\pi(-\alpha_2(t_2)+\alpha_1(t_1))}\int_{-i\infty}^{+i\infty}\frac{d\lambda}{2\pi i}\,\frac{\Gamma(\lambda-\alpha_1(t_1))\Gamma(\lambda-\alpha_2(t_2))\Gamma(-\lambda)}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))}\sin\pi(\lambda-\alpha_2(t_2))(\kappa)^{-\lambda}Q(M^2;-\lambda;t,t_1,t_2)\,, \qquad (2.4)$$

where the contour may be closed to the left [also giving the series (2.3)]. General analyticity requirements dictate that V_{12} and V_{21} have only physical M^2 singularities (they are analytic functions of κ).

As explained in (FJ), in order to achieve planar pole self-consistency it is necessary that the amplitude $\kappa^{\alpha_2(t_2)}V$ have a good FMSR of type (6) of (FJ). We now list a set of sufficient conditions on these amplitudes that guarantee good FMSR's:

I. For $M^2\geq M_0^2$, $(1/2i)\Delta_{M^2}Q$ is nonsingular in the λ plane.

II. The function Q has sufficient convergence in the λ plane to define the integral representations (2.1) and (2.4).

III. After taking the physical M^2 discontinuity of (2.1) and (2.4), the λ integrals converge sufficiently to allow the contours to be closed to the left or to the right.

We now show that I, II, and III do indeed guarantee good FMSR's. By starting with (2.1) and (2.4), taking physical M^2 discontinuities and closing λ contours to the right (assuming condition III above is satisfied) we deduce

$$(\kappa)^{\alpha_2(t_2)}\frac{1}{2i}\;\Delta_{M^2}V=\frac{\sin\pi(-\alpha_1(t_1)+\alpha_2(t_2))}{\sin\pi(-\alpha_1(t_1))}\times\frac{1}{2i}\;\Delta_{M^2}V_{21}\,. \qquad (2.5)$$

As mentioned in FJ it is well known that V_{21} by itself in general satisfies a good FMSR (since it has only a physical M^2 cut and has Regge behavior). Thus we may conclude that $\kappa^{\alpha_2(t_2)}V$ in general obeys a good FMSR. As shown in FJ, however, the good FMSR must fail at the isolated point $\alpha_1(t_1)=0$.

We observe that the dual resonance model, where Q is the specific beta function given earlier, fails to satisfy condition III. This failure is consistent with our conclusion in FJ that the dual resonance model for A_6 cannot satisfy a good FMSR.

III. A MODEL FOR A_6

We will focus our attention in this section on a specific model for A_6. It is the simplest model having Regge behavior and the correct analyticity in energy, viz., a single asymptotic power. In this case we have

$$Q=Q_*=\Gamma(-\alpha(t)+\alpha_1(t_1)+\alpha_2(t_2)-\lambda)\,g(t_1,b';t)\,g(t_2,a';t)\,(-M^2+M_0^{\,2})^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)+\lambda}\,,$$

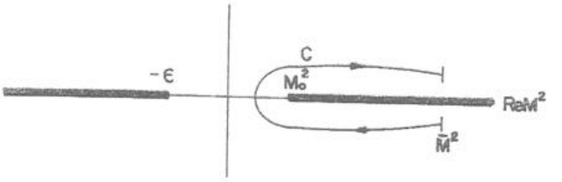

FIG. 2. Contour C for FMSR in the M^2 plane.

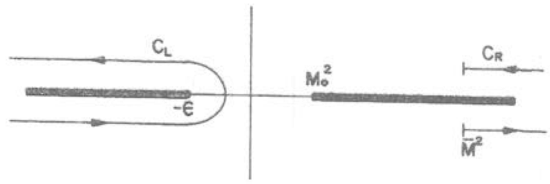

FIG. 3. Result of distorting contour C in Fig. 2.

so that the Reggeon amplitude V becomes

$$V=V_*=g(t_1,b';t)g_2(t_2,a';t)\int_{-i\infty}^{+i\infty}\frac{d\lambda\,\Gamma(\lambda-\alpha_1(t_1))\Gamma(\lambda-\alpha_2(t_2))}{2\pi i\,\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))}\,\Gamma(-\lambda)$$
$$\times(-\kappa)^{-\lambda}(-M^2+M_0^2)^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)+\lambda}\Gamma(-\alpha(t)+\alpha_1(t_1)+\alpha_2(t_2)-\lambda)$$
$$=(-M^2+M_0^2)^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)}\,\frac{\Gamma(-\alpha(t)+\alpha_1(t_1))\,\Gamma(-\alpha(t)+\alpha_2(t_2))}{\Gamma(-\alpha(t))}$$
$$\times g(t_1,b';t)g(t_2,a';t)\,{}_2F_1\left(-\alpha_1(t_1),-\alpha_2(t_2);-\alpha(t)\,\middle|\,1-\frac{M^2-M_0^2}{\kappa}\right)\,. \tag{3.1}$$

The amplitude in (3.1) gives factorizable Regge behavior in the limit $M^2\to\infty$. In the particle pole limit when $\alpha_1(t_1)$ and/or $\alpha_2(t_2)$ are equal to zero it reduces to simple amplitudes having factorizable Regge behavior. These properties as well as the explicit Steinmann decomposition may be easily verified by the reader.

It is straightforward to confirm that conditions I, II, and III of Sec. II are indeed satisfied by V_*. Thus a good FMSR must exist for $\kappa^{\alpha_2(t_2)}V_*$.

We now explicitly calculate this FMSR, that is, we evaluate the integral

$$\int_C dM^2(\kappa)^{\alpha_2(t_2)}\frac{1}{2i}\,V_*\,, \tag{3.2}$$

where the contour is shown in Fig. 2, which displays the physical cut in M^2 as well as the kinematic cut at $M^2=-\epsilon$. A convenient way to proceed is to first evaluate the Mellin transform

$$\frac{1}{2i}\int_C dM^2(-M^2+M_0^2)^{-J-1}(\kappa)^{\alpha_2(t_2)}V_*\,, \tag{3.3}$$

and then ultimately set $J=-1$.

By going to sufficiently large J we may distort the contour C into that shown in Fig. 3 (dropping justifiably the contours at infinity).

We first examine the integral over C_L in Fig. 3. This integral is just a number and, as discussed in (FJ), must vanish at $J=-1$ [since for $J=-1$ it becomes the integral over the left-hand cut in Eq. (10) of (FJ)].

Using (3.1) this integral becomes

$$\frac{1}{2i}\int_{C_L} dM^2(-M^2+M_0^2)^{-J-1}(\kappa)^{\alpha_2(t_2)}V_*$$
$$=g(t_1,b';t)g(t_2,a';t)\,\frac{\Gamma(\alpha_1(t_1)-\alpha_2(t_2))\Gamma(\alpha_2(t_2)-\alpha(t))}{\Gamma(-\alpha_2(t_2))}\;e^{-i\pi(\alpha_2(t_2)-\alpha_1(t_1))}$$
$$\times\sin\pi(\alpha_2(t_2)-\alpha_1(t_1))\int_{-\infty}^{-\epsilon}dM^2(-M^2+M_0^2)^{-J-1+\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)}(M^2-M_0^2)^{\alpha_1(t_1)}(\kappa)^{\alpha_2(t_2)-\alpha_1(t_1)}$$
$$\times\,{}_2F_1\left(-\alpha_1(t_1),\alpha_2(t_2)-\alpha(t);1-\alpha_1(t_1)+\alpha_2(t_2)\,\middle|\,\frac{\kappa}{M^2-M_0^2}\right)$$
$$=-\pi g(t_1,b';t)g(t_2,a';t)(M_0^2+\epsilon)^{\alpha(t)-\alpha_1(t_1)-J}e^{i\pi\alpha_1(t_1)}\;\frac{\Gamma(-\alpha(t)+\alpha_2(t_2))\Gamma(J-\alpha(t)+\alpha_1(t_1))\Gamma(J+1+\alpha_1(t_1))}{\Gamma(-\alpha_2(t_2))\Gamma(J+1-\alpha(t)+\alpha_1(t_1)+\alpha_2(t_2))\Gamma(J+1)}\,. \tag{3.4}$$

From (3.4) we see that the integral over C_L vanishes when analytically continued to $J=-1$, confirming that $\kappa^{\alpha_2(t_2)}V_*$ satisfies a good FMSR. Note that the J-plane singularities of (3.4) include not only the conventional poles at $J-\alpha(t)+\alpha_1(t)=0, -1, -2, \ldots$ but also the unconventional poles at $J+1+\alpha_1(t)=0,-1,-2,\ldots$. These poles give the unconventional asymptotic terms required according to FJ. If before continuing to $J=-1$, one first continues $\alpha_1(t)\to 0$, it is apparent that a *nonvanishing* constant results giving the breakdown of the FMSR at $\alpha_1(t_1)=0$ discussed in (FJ).

We now proceed to compute the FMSR (3.2). Utilizing the contour deformation of Fig. 3 and our previous result that the integral over C_L vanishes, we can write (3.2) as follows:

$$\frac{1}{2i}\int_C dM^2\kappa^{\alpha_2(t_2)}V_* = \int_{M_0^2}^{\overline{M}^2} dM^2\kappa^{\alpha_2(t_2)}\frac{1}{2i}\Delta_{M^2}V_*$$

$$= \int_{\overline{M}^2}^{\infty} dM^2\kappa^{\alpha_2(t_2)}\frac{1}{2i}\Delta_{M^2}V_*. \tag{3.5}$$

Evaluating (3.5) by means of the last integral requires a knowledge of the asymptotic behavior of $\Delta_{M^2}V_*$. Taking the discontinuity of 3.1) across the physical cut, we find

$$\frac{1}{2i}\Delta_{M^2}V_* = \pi g(t_1,b';t)g(t_2,a';t)\frac{(M^2-M_0^2)^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)}}{\Gamma(1+\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2))}$$

$$\times {}_2F_1\left(-\alpha_1(t_1), -\alpha_2(t_2); 1+\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)\,\middle|\,\frac{M^2-M_0^2}{\kappa}\right), \tag{3.6}$$

which has the asymptotic behavior

$$\frac{1}{2i}\Delta_{M^2}V_* \underset{M^2\to\infty}{\sim} \pi g(t_1,b';t)g(t_2,a';t)$$

$$\times\left\{\frac{\Gamma(1+\alpha(t))(M^2-M_0^2)^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)}}{\Gamma(1+\alpha(t)-\alpha_1(t_1))\Gamma(1+\alpha(t)-\alpha_2(t_2))}\left[1+O\left(\frac{1}{M^2}\right)\right]\right.$$

$$\left.+\frac{\Gamma(-1-\alpha(t))(M_0^2+\epsilon)^{1+\alpha(t)}}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))}(M^2-M_0^2)^{-1-\alpha_1(t_1)-\alpha_2(t_2)}\left[1+O\left(\frac{1}{M^2}\right)\right]\right\}. \tag{3.7}$$

The second asymptotic term in (3.7) corresponds to the unconventional terms discussed in FJ that are required if the amplitude is to satisfy a good FMSR. It also represents the asymptotic behavior corresponding to the poles in the Mellin transform at $J+1+\alpha_1(t_1)=0,-1,-2,$ mentioned earlier. Note that although the second term of (3.7) appears to be singular at $\alpha(t)=0$, this singularity is in fact canceled by subdominant terms in M^2 [since the hypergeometric function in (3.6) is nonsingular at $\alpha(t)=0,\pm1,\pm2,\ldots$ there must be such cancellations in the asymptotic series (3.7)].

We now compute the FMSR using (3.5) and (3.7):

$$\int_{\overline{M}^2}^{\infty} dM^2(\kappa)^{\alpha_2(t_2)}\frac{1}{2i}\Delta_{M^2}V_* = g(t_1,b';t)g(t_2,a';t)$$

$$\times\left\{\frac{\Gamma(1+\alpha(t))(\overline{M}^2)^{\alpha(t)-\alpha_1(t_1)+1}}{\Gamma(1+\alpha(t)-\alpha_1(t_1))\Gamma(1+\alpha(t)-\alpha_2(t_2))(1+\alpha(t)-\alpha_1(t))}\left[1+O\left(\frac{1}{\overline{M}^2}\right)\right]\right.$$

$$\left.+\frac{\Gamma(-1-\alpha(t))(M_0^2+\epsilon)^{1+\alpha(t)}(\overline{M}^2)^{-\alpha_1(t_1)}}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))(-\alpha_1(t_1))}\left[1+O\left(\frac{1}{\overline{M}^2}\right)\right]\right\}. \tag{3.8}$$

Even though, as we have shown, the left-hand cut in Fig. 3 gives no contribution to the FMSR (3.8), nonetheless at $\alpha_1(t_1)=0$ (3.8) develops a constant term independent of $\overline{M}^2$, as it should.

IV. DISCUSSION

We have given here a simple model for A_6 of Fig. 1 that satisfies a good FMSR. Furthermore, we have established general sufficiency conditions for A_6 to satisfy good FMSR's. These conditions complement the necessary conditions established in (FJ). We emphasize that the dual resonance model does not satisfy these conditions and therefore would generate Regge cuts in the planar S matrix.

It is nevertheless instructive to compare our simple model for A_6 (3.1) with its dual resonance model counterpart. In many respects the unconventional terms in our simple model play a role analogous to the effect of the β trajectory[4] in the dual resonance model. For example, consider the dual analog of (3.8):

$$\frac{1}{2i}\Delta_{M^2}V_{\text{DRM}} \underset{M^2\to\infty}{\sim} \pi\gamma^2 \left\{\frac{\Gamma(1+\alpha(t))(M^2)^{\alpha(t)-\alpha_1(t_1)-\alpha_2(t_2)}}{\Gamma(1+\alpha(t)-\alpha_1(t_1))\Gamma(1+\alpha(t)-\alpha_2(t_2))}\left[1+O\left(\frac{1}{M^2}\right)\right] + \frac{\Gamma(-1-\beta(t))e^{i\pi\beta(t)}2^{-\beta(t)-2}(M^2)^{\beta(t)-\alpha_1(t_1)-\alpha_2(t_2)}}{\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))}\left[1+O\left(\frac{1}{M^2}\right)\right]\right\}, \tag{4.1}$$

where $\beta(t)=\frac{1}{2}\alpha(t)-\frac{1}{2}$. One sees that the β-trajectory term serves to keep the amplitude finite at $\alpha(t)=-1$, dominates the amplitude for $\alpha(t)<-1$, and vanishes when one of the external Reggeon legs is continued to the particle pole.

We see from (3.8) that the second (unconventional) asymptotic term for our simple model serves in much the same role as the β-trajectory term in (4.1). The unconventional term in (3.8) keeps $\Delta_{M^2}A_6$ finite at $\alpha(t)=-1$; it dominates for $\alpha(t)<-1$, and vanishes for $\alpha_1(t_1)=0$ or $\alpha_2(t_2)=0$.

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