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Existence of good finite-mass sum rules and the planar pole bootstrap*

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The crucial role of "good" finite-mass sum rules in the planar pole bootstrap problem is emphasized. We establish a necessary condition for certain Reggeon amplitudes to satisfy good finite-mass sum rules: the amplitudes must have an unconventional term in their asymptotic behavior. The dual resonance model fails to satisfy this condition.

Currently topological expansion[1] (TE) or dual unitarization[2] provides an ambitious framework for confronting strong-interaction dynamics. The first term in the TE is the planar approximation and higher-order terms come from nonplanar corrections. It has been conjectured but not proved that planar unitarity is consistent with planar amplitudes having Regge poles only and no Regge cuts. Such a planar pole bootstrap is not ruled out *a priori* since planar amplitudes lack a third double-spectral function which is known to make Regge cuts inescapable. Establishing the existence of planar self-consistency is of fundamental importance since the planar approximation bases the TE program.

We shall first show, as previous work has suggested,[3-5] that the existence of a planar pole bootstrap requires that certain Reggeon amplitudes satisfy "good" finite-mass sum rules (FMSR's). In particular, we investigate under what circumstances such FMSR's are obeyed in the simplest nontrivial case involving the amplitude A_6 of Fig. 1. We establish a necessary condition for A_6 to satisfy a good FMSR: The amplitude must have an unconventional term in its asymptotic behavior. From our results, it follows that the dual resonance model does *not* possess such a good FMSR. If the results of this paper generalize to amplitudes with a larger number of external Reggeons, then a planar pole bootstrap cannot be based in detail on asymptotic behavior suggested by the dual resonance model. In the following paper we display a simple model for A_6 which *does* possess a good FMSR and hence may serve as a candidate for a self-consistent planar amplitude.

To illustrate how FMSR's come into the planar pole bootstrap problem we focus on the simplest nontrivial unitarity equations which must be satisfied. Figure 2 depicts the unitarity relation for the four-line amplitude with one external Reggeon. [The simpler unitarity relation for the four-line amplitude with no external Reggeons, i.e., $\alpha_2(t_2)=0$ in Fig. 2, can readily be shown to be self-consistent with Regge poles alone[6] and will be discussed as a limiting case of Fig. 2.]

The unitarity sum for the two-bubble diagrams on the right-side of Fig. 2 must be carefully specified to avoid double counting. We use the following simple procedure for specifying the summation: Those particles with rapidities lying in the first half of the total rapidity interval are to be assigned to the left-hand bubble, those in the second half to the right-hand bubble. This procedure means, of course, that the Regge exchanges $\alpha_1(t_1)$ and $\alpha_1'(t_1')$ in Fig. 2 will be used even to describe reactions where small rapidity gaps separate the left and right bubbles. Thus we shall assume in Fig. 2 that a summation is carried out over as many Regge-pole exchanges $\alpha_1(t_1)$ and $\alpha_1'(t_1')$ as are needed to obtain an accurate expression for the amplitudes.

It turns out that the points we wish to consider here are amply illustrated by letting $\alpha_1'(t_1')=0$ in Fig. 2 corresponding to the exchange of a spin-zero particle. This leads to the evaluation of the graph in Fig. 3. The right-hand bubble in Fig. 3 is simply related to A_6 in Fig. 1 and the properties of A_6 are better understood than those of the cor-

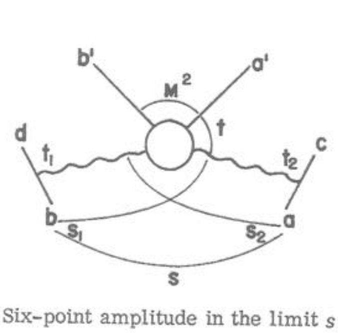

FIG. 1. Six-point amplitude in the limit $s_1, s_2 \to \infty$.

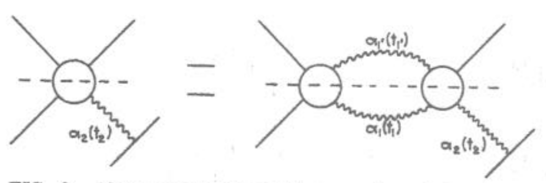

FIG. 2. Planar unitarity with one external Reggeon.

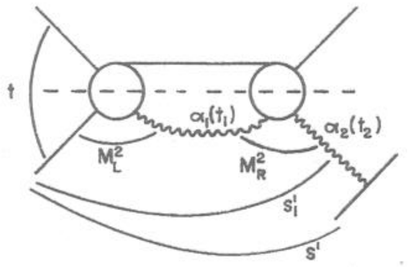

FIG. 3. Planar unitarity sum with $\alpha_1'(t_1')=0$.

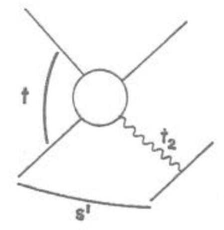

FIG. 4. Five-point amplitude relevant to three-particle, one-Reggeon amplitude.

responding amplitude in Fig. 2 which involves three Reggeons.

We now proceed to write down the unitarity sum of Fig. 3 in accord with the previous discussion. Assuming factorization for Regge-pole exchanges we see that this sum can be computed through knowledge of the A_6 amplitude in the limit shown in Fig. 1 and of the A_5 amplitude in the limit of Fig. 4. We have, in the notation of Fig. 1,

$$A_6=\beta_1(t_1)\beta_2(t_2)\Gamma(-\alpha_1(t_1))\times\Gamma(-\alpha_2(t_2))(-s_1)^{\alpha_1(t_1)}(-s_2)^{\alpha_2(t_2)}V, \quad (1)$$

where $\alpha_1(t_1)$ and $\alpha_2(t_2)$ are the trajectory functions, the β's are residue functions, and V is the two-Reggeon–two-particle amplitude. Similarly for A_5 of Fig. 4 we have

$$A_5=\beta_2(t_2)\Gamma(-\alpha_2(t_2))(-s')^{\alpha_2(t_2)}W, \quad (2)$$

where W is the four-line amplitude with one external Reggeon.

Combining (1) and (2) above and using factorization, we arrive at the following expression for the unitarity sum in Fig. 2, isolating the contribution of Fig. 3 (using the standard assumption of simple factorizable phase space):

$$\frac{1}{2i}\Delta_{s_1'}A_5(s',s_1',\ldots)=(-s')^{\alpha_2(t_2)}\int d\Omega_1(s_1')^{\alpha_1(t_1)-\alpha_2(t_2)-1}\times\int_{M_0^2}^{(s_1')^{1/2}}dM_L^2\left(\frac{1}{2i}\Delta_{M_L^2}W\right)\int_{M_0^2}^{(s_1')^{1/2}}dM_R^2(\kappa)^{\alpha_2(t_2)}\left(\frac{1}{2i}\Delta_{M_R^2}V\right)\times\Gamma(-\alpha_1(t_1))\Gamma(-\alpha_2(t_2))\beta_2(t_2)+\cdots, \quad (3)$$

where $d\Omega_1$ is the phase-space integration including the integration over t_1 and

$$\kappa=s_1s_2/s=M^2+\epsilon$$

with ϵ a function of the fixed momentum-transfer variables. The upper limits in the M_L^2 and M_R^2 integrations reflect the rapidity division mentioned earlier.

We see in (3) that the integration over t_1 apparently produces Regge cut behavior at large s_1' as would be suggested by the diagram in Fig. 3. However, we now indicate that the apparent cut behavior is eliminated if the amplitudes W and V satisfy certain finite-mass sum rules. First, we note that W possesses a Regge-pole expansion, the leading term of which is given by

$$W\approx\gamma(-M_L^2)^{\alpha(t)-\alpha_1(t_1)}, \quad (4)$$

where γ represents the appropriate residue functions. Owing to our basic assumption of planarity, W possesses only a right-hand physical cut. Amplitudes that have Regge behavior with only a right-hand cut are well known to satisfy "good" FMSR's of the following form:

$$\frac{1}{2i}\int_{M_0^2}^{\overline{M}^2}dM_L^2\Delta_{M_L^2}W=\sum_{\omega}\frac{(\overline{M}^2)^{\omega+1}D(\omega,t,\ldots)}{\omega+1}, \quad (5)$$

where the ω summation is over the Regge powers present in W. For the amplitude W, the leading value of ω is $\omega=\alpha(t)-\alpha_1(t_1)$. Setting $\overline{M}^2=(s_1')^{1/2}$ in (5) we see that each term in the sum contains the t_1-dependent factor $(s_1')^{-\alpha_1(t_1)/2}$. Now if each term in the FMSR for V in (3) has a similar factor, we see that there will be no Regge-cut behavior arising in the unitarity sum (3).

We first observe that when $\alpha_2(t_2)=0$ corresponding to the unitarity sum of Fig. 5, we have $V=W$ and the FMSR over W in (3) is the same as (5) so that the Regge cut behavior is canceled. Furthermore, the leading behavior of (3) is proportional to $(s_1')^{\alpha(t)}$, which is the required self-consistent be-

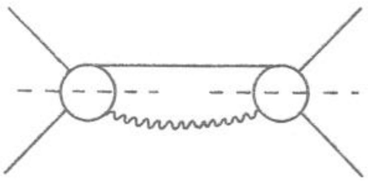

FIG. 5. Planar unitarity sum with $\alpha_2(t_2)=0$.

havior, i.e., the Regge-pole bootstrap is satisfied. This result has been known for some time.[6]

The discussion of the planar bootstrap when $\alpha_2(t_2)\neq 0$ is much more subtle. In this case in order to generate a pure pole output for the unitarity sum of Fig. 3, the required FMSR is of the type

$$\frac{1}{2i}\int_{M_0^2}^{\bar{M}^2} dM^2(\kappa)^{\alpha_2(t_2)}\Delta_{M^2}V = \sum_{\omega}\frac{(\bar{M}^2)^{\omega+1}\bar{D}(\omega,\alpha_1,\alpha_2,\epsilon,t,t_1,t_2)}{\omega+1}, \quad (6)$$

where ω is of the form $\omega=\omega'-\alpha_1(t_1)$ with ω' independent of t_1. The difficulty in establishing the FMSR (6) arises because, although V has Regge behavior $[\omega=\alpha(t)-\alpha_1(t_1)$ for the leading behavior$]$, the integrand in (6) has a kinetic cut at $\kappa=M^2+\epsilon=0$ in addition to the physical threshold at $M^2=M_0^2$, making the usual proofs for FMSR's invalid. The presence of such kinematic cuts does not violate the planarity of the basic amplitudes but merely reflects the phase-space restrictions of the unitarity integral.

We first write down the Steinmann decomposition for the amplitude V (see also Fig. 1):

$$V=(-\kappa)^{-\alpha_1(t_1)}V_{12}(M^2,\kappa;t,t_1,t_2) + (-\kappa)^{-\alpha_2(t_2)}V_{21}(M^2,\kappa;t,t_1,t_2). \quad (7)$$

Note that we include here only those two Steinmann pieces of the six-point function which possess a physical M^2 discontinuity as it is these pieces which enter into a unitarity summation. As is customary, two helicity variables have been set equal to zero in (7) reducing the total number of variables in A_6 from eight to six. Both V_{12} and V_{21} have only threshold cuts in M^2 and possess the following Regge behavior:

$$V_{12}\underset{M^2\to\infty}{\sim}(-M^2)^{\alpha(t)-\alpha_2(t_2)}g(t,t_1)g(t,t_2),$$
$$V_{21}\underset{M^2\to\infty}{\sim}(-M^2)^{\alpha(t)-\alpha_1(t_1)}g(t,t_1)g(t,t_2), \quad (8)$$

where the g's are the appropriate Regge residues. Both V_{12} and V_{21} are analytic in the κ variable. Inserting expression (7) into (6) we see that with the weighting factor $(+\kappa)^{\alpha_2(t_2)}$, one piece of the integrand in (7) is just V_{21} which has by itself a good FMSR of the type (5). So the question becomes: Under what circumstances will the remaining piece coming from V_{12} have a good FMSR? This part of the integral in (6) is of the form

$$\frac{1}{2i}\int_{M_0^2}^{\bar{M}^2} dM^2(+\kappa)^{\alpha_2(t_2)-\alpha_1(t_1)}\Delta_{M^2}V_{12}(M^2,\kappa;t,t_1,t_2). \quad (9)$$

Recalling that $\kappa=M^2+\epsilon$, we see that (9) corresponds to an FMSR integral for an amplitude with Regge behavior which, in addition to a right-hand physical cut starting at M_0^2, has a kinematic left-hand cut starting at $M^2=-\epsilon$. In general such amplitudes need *not* obey good FMSR's and the integral in (9) will have the form

$$\int_{M_0^2}^{\bar{M}^2} dM^2(\kappa)^{\alpha_2(t_2)-\alpha_1(t_1)}\frac{1}{2i}\Delta_{M^2}V_{12}(M^2,\ldots)=\sum_{\omega'}\frac{(\bar{M}^2)^{\omega'-\alpha_1(t_1)+1}\bar{D}'(\omega',\ldots)}{\omega'-\alpha_1(t_1)+1} + \int_{-\infty}^{-\epsilon} dM^2[V_{12}(M^2,\ldots)]\frac{1}{2i}\Delta_{M^2}(M^2+\epsilon)^{\alpha_2(t_2)-\alpha_1(t_1)}, \quad (10)$$

where the second term in (10) is a constant corresponding to the integral of the discontinuity over the left-hand cut. (This integral, of course, may have to be defined by analytic continuation.)

If the amplitude in (10) is to have a good FMSR, we see that either (i) it must have no left-hand cut or (ii) the *integral* over the left-hand cut must vanish, in which case the amplitude superconverges separately on the right- and left-hand cuts. Since the Reggeon amplitude that we are considering definitely possesses a left-hand cut only option (ii) is open. We stress that the presence of a left-hand cut (coming from κ dependence) does *not* automatically preclude the existence of a good FMSR.

We now show that if good FMSR's exist at all then there must be an unconventional term in the Regge asymptotic behavior of $(1/2i)\Delta V$.

First we note that a good FMSR *must* fail to exist when $\alpha_1(t_1)=0$. In this case V_{12} becomes just the Reggeon amplitude with one external Reggeon, shown in Fig. 4 (note that the function V_{21} vanishes at $\alpha_1 = 0$ but V_{12} does not). Whenever $\alpha_1 = n$, the function V_{12} is a polynomial of order n in κ. Angular momentum considerations show that when $\alpha_1(t_1)$ is an integer the κ behavior in V_{12} is a polynomial limited in degree by the spin $\alpha_1(t_1)$. Thus when $\alpha_1(t_1) = 0$, V_{12} has *no* dependence on κ. Accordingly, for $\alpha_1(t_1) = 0$ the integral over the left-hand cut in (10) cannot vanish for it becomes of the form

$$\int_{-\infty}^{-\epsilon} dM^2[V_{12}(M^2, \dots)]\frac{1}{2i}\,\Delta_{M^2}[(M^2+\epsilon)^{\alpha_2(t_2)}], \qquad (11)$$

where V_{12} is now independent of ϵ. The integral in (11) cannot vanish as an identity in ϵ if V_{12} is independent of ϵ. An example of this general truth can be seen by setting $\alpha_2(t_2) = -1$ in (11); the entire integral comes from a pole whose residue is proportional to V_{12} $(M^2 = -\epsilon, \dots)$ which cannot vanish for arbitrary ϵ unless V_{12} is identically zero. Thus there is no good FMSR in Eq. (10) when $\alpha_1(t_1) = 0$.

We may conclude that if the amplitude in (10) satisfies a good FMSR,

$$\int_{M_0^2}^{\overline{M}^2} dM^2(\kappa)^{\alpha_2(t_2)-\alpha_1(t_1)}\left[\frac{1}{2i}\,\Delta_{M^2}V_{12}(M^2, \kappa; t, t_1, t_2)\right] = \sum_{\omega'} \frac{(\overline{M}^2)^{\omega'-\alpha_1(t_1)+1}\overline{\overline{D}}(\omega', \dots)}{\omega' - \alpha_1(t_1)+1}, \qquad (12)$$

then this FMSR must fail, i.e., develop a constant term when $\alpha_1(t_1) = 0$. This can happen only if (12) includes a term where $\omega' = -1$ and such that $\overline{\overline{D}}$ has a linear zero at $\alpha_1(t_1) = 0$.

This then establishes a *necessary* condition on the Reggeon amplitude V in order that it satisfy a good FMSR: The physical M^2 discontinuity of V in (7) must have an asymptotic term of the form $(M^2)^{-1-\alpha_1(t_1)-\alpha_2(t_2)}$.

The Reggeon amplitude corresponding to the A_6 of the dual resonance model does *not* possess the unconventional asymptotic term required for a good FMSR. Whereas recent studies of A_6 in the dual resonance model by Hoyer *et al.*[7] have shown the presence of new-type Regge asymptotic behavior, the so-called β trajectories, this asymptotic behavior does not correspond to the unconventional type discussed here. Thus, even though planar dual amplitudes pass some of the consistency requirements of planar unitarity[6] [for example, the unitarity requirement shown in Fig. 2 but with $\alpha_2(t_2)$ continued to a particle leg], nevertheless the dual amplitude for A_6 lacks a necessary ingredient for planar self-consistency, namely a "good" FMSR.

In the following paper, we consider the general question of finding Reggeon amplitudes V satisfying good FMSR's. We present there a simple model amplitude for V which satisfies a good FMSR and we also give a sufficiency condition for the existence of good FMSR's. Whether such amplitudes can be used as the basis for a satisfactory self-consistent planar bootstrap is under study.

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