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BUILDING A CYLINDER IN THE TOPOLOGICAL EXPANSION *

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Usual cylinder constructions neglect planar analyticity and the resulting cylinder is contaminated by j-plane cuts. Beginning with input planar amplitudes that are manifestly consistent with a pole-to-pole planar bootstrap, a three-dimensional cylinder equation is formulated, incorporating planar analyticity. The output cylinder is now free of j-plane cuts. The bare pomeron is a simple pole with trajectory intercept slightly above one.

1. Introduction

In this paper I address myself to building a cylinder in the topological expansion [1] (TE) or dual unitarization [2]. While there have appeared in the literature several studies [2–8] of the cylinder correction to the planar S-matrix, these analyses do not take full account of the analyticity properties of the planar amplitudes that are sewn together to construct the cylinder. Specifically, these works neglect in their cylinder the subtleties of the low-energy behavior of planar amplitudes and incorporate only the planar high-energy Regge asymptotic form.

Study of planar unitarity has shown that approximating planar amplitudes by simply their high-energy Regge asymptotic form is inadequate if planar self-consistency is to be attained. The correct inclusion of low-energy planar amplitudes is vital in achieving a pure pole-to-pole bootstrap without j-plane cuts [9–11]. Since the *same* planar amplitudes that are consistent with the planar bootstrap ought to be used in cylinder building, there is no justification for sewing together only high-energy planar amplitudes.

In this paper, I build a cylinder using input planar amplitudes manifestly consistent with planar unitarity. Despite the complication of maintaining full planar analyticity, the output cylinder is amazingly simple. It consists of two j-plane poles and *no cuts*! The leading singularity, the bare pomeron, has a trajectory function whose intercept is slightly above unity. The subdominant cylinder singularity exactly cancels with the high-energy singlet planar exchange, consistent with the "fextinc-

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tion" or "pomeron-f identity" advocated by Chew and Rosenzweig [4] and Schmic and Sorensen [5]. In the one-dimensional limit where transverse directions are suppressed, this cylinder becomes equivalent to the diagram-counting model of Huan-Lee [12] with a unit-intercept bare pomeron.

The structure of this paper is as follows. Sect. 2 reviews the essential features of self-consistent planar amplitudes that will subsequently be important in erecting a cylinder. Sect. 3 analyzes the one-twist contribution to the cylinder. The result is *not* what is gotten when, as is usually done, planar analyticity is neglected. Subsect 4.1 contains a cylinder integral equation and its solution. Both the j-plane integral equation and the solution differ significantly from the commonly assumed forms which are presented in subsect. 4.2. Sect. 5 presents some remarks and conclusions.

2. Properties of planar reggeon amplitudes

This section describes the essential features of self-consistent planar reggeon amplitudes [11]. These *same* reggeon amplitudes will be used later on in cylinder building.

The planar reggeon + reggeon → reggeon + reggeon amplitude shown in fig. 1 ma be extracted from the appropriate eight-point function as in subsect. 2.1 and table of ref. [11]. To some extent the notation of ref. [11] will be followed. The four-reggeon amplitude of fig. 1 is denoted $A_{R_1R_2}(M^2, M_\perp^2; t; t_1^\pm, t_2^\pm)$, where $M_\perp^2 = M^2 +$ with $P_\perp^2$ a non-negative function of the fixed momentum-transfer variables $t, t_1^\pm, t_2^\pm$. This $M_\perp^2$ dependence is associated with a kinematic (left-hand) cut at $M^2 = -P_\perp^2$ that is present in the amplitude in addition to the (right-hand) physical M^2 singularities. Note that rapidity-variable forms of reggeon amplitudes implicitly neglect the impo tant distinction between M^2 and $M_\perp^2$ variables.

Planar (actually, ordered) unitarity severely constrains the planar reggeon amplitudes. One such unitarity equation is shown symbolically in fig. 2 and may be written [11]

$$\frac{1}{2i}\,\Delta A_{R_1R_3}(s_{23}, s_{23}^\perp; t; t_1^\pm; t_3^\pm)\,\theta(s_{23} - \bar{s})$$

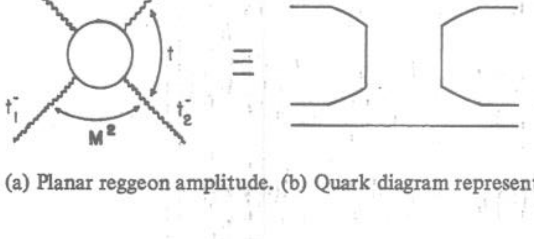

Fig. 1. (a) Planar reggeon amplitude. (b) Quark diagram representation.

$$= \theta(s_{23} - \bar{s}) N \int d\phi_2 \eta_2 s_{23}^{\alpha_{c,2} - \alpha_{c,1} - \alpha_{c,3} - 2} \int_0^{s_{23}} ds_2 \int_0^{s_{23}} ds_3$$

$$\times \theta(\bar{s} - s_2)\, \theta(\gamma s_{23} - \bar{s} s_3) (s_2^{\perp})^{\alpha_{c,1}+1}$$

$$\times \frac{1}{2i} \Delta A_{R_1 R_2}(s_2, s_2^{\perp}; t; t_1^{\pm}, t_2^{\pm}) (s_3^{\perp})^{\alpha_{c,3}+1} \frac{1}{2i} \Delta A_{R_2 R_3}(s_3, s_3^{\perp}; t; t_2^{\pm}, t_3^{\pm}) , \quad (2.1)$$

where $\alpha(t_i)$ is a trajectory function and

$$\alpha_{c,k} \equiv \alpha(t_k^+) + \alpha(t_k^-) - 1 ,$$

$$\eta_2 \equiv \Gamma(-\alpha(t_2^-))\, \Gamma(-\alpha(t_2^+)) \cos \pi [\alpha(t_2^+) - \alpha(t_2^-)] ,$$

$$d\phi_2 \equiv \frac{1}{16\pi^4} \frac{dt_2^+ dt_2^- \theta(-\lambda(t, t_2^+, t_2^-))}{[-\lambda(t, t_2^+, t_2^-)]^{1/2}} . \quad (2.2)$$

λ is the usual triangle function. The discontinuities across the reggeon amplitudes are taken only across the physical singularities. The N factor is due to the assumed internal SU(N) symmetry. The lower limit of integration over the s_2 and s_3 blobs actually begins at the physical threshold M_0^2 which is already built into the reggeon amplitudes $A_{R_1R_2}(s_2, s_2^{\perp} \ldots)$ and $A_{R_2R_3}(s_3, s_3^{\perp}, \ldots)$. The θ-function constraints in (2.1) have been chosen to strictly avoid double counting [10,11,13] of intermediate-state particle configurations in the unitarity sum. The γ factor is just a scale factor.

Explicit realization (i.e. models) of planar reggeon amplitudes that meet not only

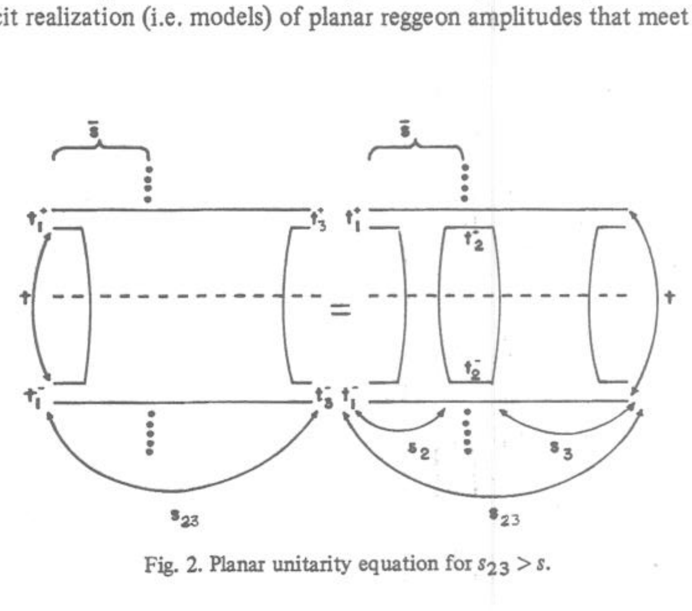

Fig. 2. Planar unitarity equation for $s_{23} > s$.

the demands of analyticity but also unitarity requirements such as (2.1) are dealt with elsewhere [14–16]. For the present purposes, it will suffice to focus only on certain essential properties of self-consistent planar amplitudes.

The amplitude $A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2, ...)$ displays Regge asymptotic behavior for sufficiently large M^2. In particular

$$\frac{1}{2i}\,\Delta A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2; ...)$$

$$\xrightarrow{\text{large } M^2} \frac{\pi}{\Gamma(\alpha(t)+1)}\, g(t; t_1^\pm)\, g(t; t_2^\pm)(M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2} + \dots\,, \tag{2.3}$$

where the g's are triple-Regge couplings.

To achieve a pole-to-pole bootstrap uncontaminated by Regge cuts, reggeon amplitudes must satisfy certain good finite-mass sum rules [9–11,14] (FMSR):

$$\int_0^Z \mathrm{d}M^2\,(M_1^2)^{\alpha_{c,1}+1}\,\frac{1}{2i}\,\Delta A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2; t; t_1^\pm, t_2^\pm)$$

$$= \frac{\pi}{\Gamma(\alpha(t)+1)}\, g(t; t_1^\pm)\, g(t; t_2^\pm)\,\frac{Z^{\alpha(t)-\alpha_{c,2}}}{\alpha(t)-\alpha_{c,2}} + \dots\,, \tag{2.4}$$

$$\int_0^Z \mathrm{d}M^2\,(M_1^2)^{\alpha_{c,2}+1}\,\frac{1}{2i}\,\Delta A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2; t; t_1^\pm, t_2^\pm)$$

$$= \frac{\pi}{\Gamma(\alpha(t)+1)}\, g(t; t_1^\pm)\, g(t; t_2^\pm)\,\frac{Z^{\alpha(t)-\alpha_{c,1}}}{\alpha(t)-\alpha_{c,1}} + \dots\,. \tag{2.5}$$

The crucial point is that there are no Z-independent terms on the right-hand side of (2.4) or (2.5). This simply reflects the vanishing of contour integrals encircling the kinematic cut of $(M_1^2)^{\alpha_{c,1}+1}A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2, ...)$ or $(M_1^2)^{\alpha_{c,2}+1}A_{\mathrm{R}_1\mathrm{R}_2}(M^2, M_1^2, ...)$.

Using (2.3), (2.4), and (2.5) in the unitarity equation (2.1), it is straightforward to show planar pole self-consistency is indeed achieved provided the usual bootstrap condition [17] is satisfied:

$$1 = \frac{\pi}{\Gamma(\alpha(t)+1)}\, N \int \mathrm{d}\phi_2\, g^2(t; t_2^\pm)\,\frac{\gamma^{\alpha(t)-\alpha_{c,2}}}{[\alpha(t)-\alpha_{c,2}]^2}\,\eta_2\,. \tag{2.6}$$

This non-linear normalization of the triple-Regge couplings will be used in sect. 4 in estimating cylinder parameters.

A few additional comments are in order. Recent work [14–16] has shown that there are undoubtedly in addition, unconventional terms on the right-hand side of (2.3) and the good FMSR (2.4), (2.5). For example, the general analytic requirement that, when the reggeons R_2 and R_2' are continued to particle poles (i.e., $\alpha(t_2)$, $\alpha(t_2') \to 0$), the good FMSR (2.4) must fail (i.e. develop a constant, Z-independent piece) necessitates an unconventional $(M^2)^{-1-\alpha_{c,1}-\alpha_{c,2}-2}$ piece in the asymptotic form of $(1/2i)\,\Delta_{M^2} A_{R_1R_2}(M^2, M_1^2, ...)$. Such unconventional terms have the virtue of preventing unwanted singularities in $\Delta_{M^2} A_{R_1R_2}(M^2, M_1^2, ...)$ at the "trouble spots" $\alpha(t) = -1, -2, -3, ...$. They also decouple from the four-external particle amplitude. To avoid possible complications coming from these unconventional terms, we simply keep $\alpha(t)$ sufficiently greater than -1 so that the unconventional terms are subdominant and may be safely ignored.

In an interesting paper [18] Kwiecinski and Sakai argue against Regge-cut cancellations when finite-size clusters are summed over in planar unitarity; however, their conclusion follows from an analysis of particular j-plane unitarity equations whose general validity is suspect. If planar unitarity is formulated carefully in the energy plane as in ref. [11] or eq. (2.1), maintaining the important distinction between s_i and $s_i^\perp$ variables, then the j-plane projection *via* a Mellin transform does *not* generate the diagonalized j-plane equations of Kwiecinski and Sakai. Thus their conclusion applies only to those approximate formulations of planar unitarity, such as rapidity-variable formulations, that neglect the difference between S_i and $S_i^\perp$ variables. More precise formulations such as eq. (2.1) indeed allow a self-consistent bootstrap (uncontaminated by Regge cuts) if the planar amplitudes are Regge behaved and obey certain good FMSR. Furthermore, the requirement of these good FMSR (2.4) and (2.5) does not force undesirable threshold singularities on planar reggeon amplitudes, as is evident from the explicit example in ref. [15].

3. The one-twist piece of the cylinder

Symbolically, a cylinder may be represented as in figs. 3, 4, by a sum of planar amplitudes sewn together in a particular non-planar fashion [2]. The difficulty in

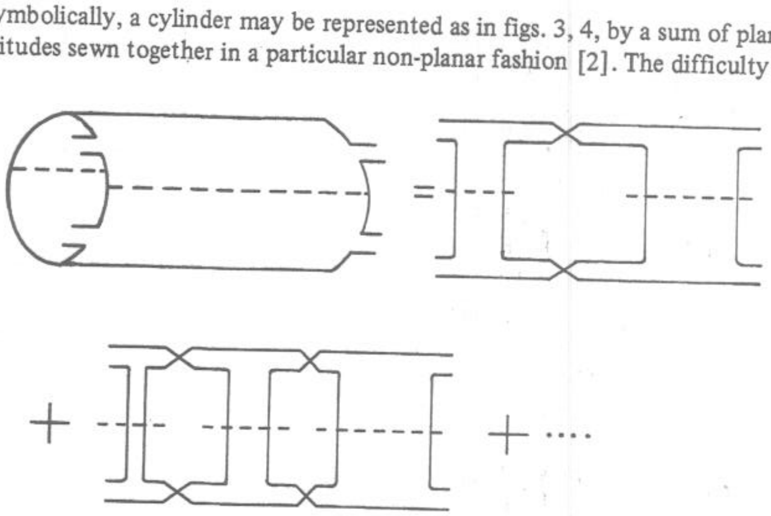

Fig. 3. A cylinder in series form.

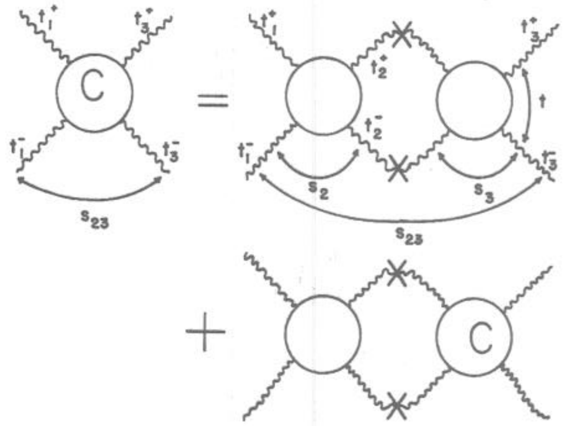

Fig. 4. A cylinder equation.

actually carrying out this summation lies in making sure that all the intermediate-state particle configurations have been included. There are no "double counting" subtleties [10,11,13] as in the planar bootstrap since in the cylinder construction twisted propagators separate the sewn-together planar amplitudes. However, the contributions from low-energy planar amplitudes may be overlooked unless care is taken. To make this potential pitfall very clear, I will first analyze the contribution of the first term in the series of fig. 3, namely the one-twist term, hereafter denoted $C^{(1)}_{R_1R_3}(s_{23}; t; t_1^\pm, t_3^\pm)$.

Consider this one-twist term at sufficiently large energy, say

$$s_{23} > \omega > \gamma \,. \tag{3.1}$$

Then

$$\begin{aligned} &C^{(1)}_{R_1R_3}(s_{23}; t; t_1^\pm, t_3^\pm)\, \theta(s_{23} - \omega) \\ &\quad = \theta(s_{23} - \omega) \int \mathrm{d}\phi_2 \xi_2 (s_{23})^{\alpha_{c,2} - \alpha_{c,1} - \alpha_{c,3} - 2} \int_0^{s_{23}} \mathrm{d}s_2 \int_0^{s_{23}} \mathrm{d}s_3\, \theta(\gamma s_{23} - s_2 s_3) \\ &\quad \times (s_2^\perp)^{\alpha_{c,1}+1} \frac{1}{2i} \Delta A_{R_1R_2}(s_2, s_2^\perp; t; t_1^\pm, t_2^\pm) \\ &\quad \times (s_3^\perp)^{\alpha_{c,3}+1} \frac{1}{2i} \Delta A_{R_2R_3}(s_3, s_3^\perp; t; t_2^\pm, t_3^\pm) \,, \end{aligned} \tag{3.2}$$

where

$$\xi_2 \equiv \Gamma(-\alpha(t_2^+))\, \Gamma(-\alpha(t_2^-)) \,. \tag{3.3}$$

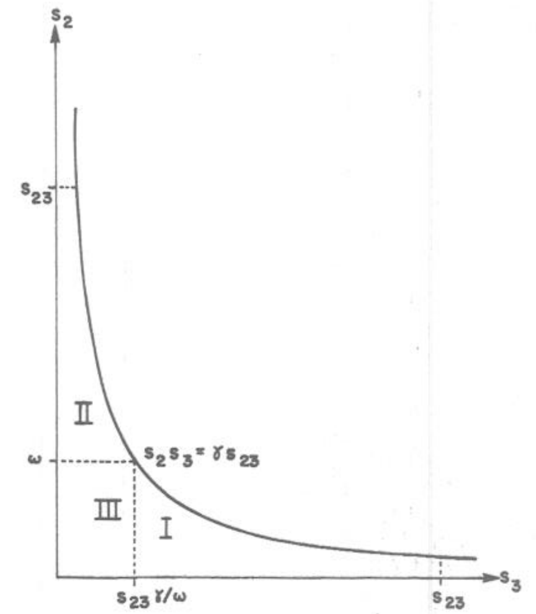

Fig. 5. Phase-space division in eqs. (3.4a–c).

Note that: (i) there is no factor of N since there is no closed quark loop, (ii) the exchanged reggeon is twisted and so has a constant phase, (iii) there is no need to impose a fixed dividing line between the two blobs to prevent multiple counting.

The phase space in (3.2) may be visualized as a sum of three contributions depending on the relative orientation of the two planar blobs with respect to the minimum available energy ω. Referring to fig. 5 these integration ranges are given by:

$$\text{(I)}\quad \theta(s_{23}-\omega)\int \mathrm{d}\phi_2 \int_{\gamma s_{23}/\omega}^{s_{23}} \mathrm{d}s_3 \int_0^{\gamma s_{23}/s_3} \mathrm{d}s_2 \ldots , \tag{3.4a}$$

$$\text{(II)}\quad \theta(s_{23}-\omega)\int \mathrm{d}\phi_2 \int_{\omega}^{s_{23}} \mathrm{d}s_2 \int_0^{\gamma s_{23}/s_2} \mathrm{d}s_3 \ldots , \tag{3.4b}$$

$$\text{(III)}\quad \theta(s_{23}-\omega)\int \mathrm{d}\phi_2 \int_0^{\omega} \mathrm{d}s_2 \int_0^{\gamma s_{23}/\omega} \mathrm{d}s_3 \ldots . \tag{3.4c}$$

Viewed this way, the phase space is seen to incorporate all the intermediate-state particle configurations when there is no overlap between particles from the two different blobs.

The integrals over the missing masses in (3.4a,b,c) are either FMSR of the type (2.4) and (2.5) or integrals over the high-energy part of planar amplitudes (in which case the high-energy planar form (2.3) may be safely used). For example, for (3.4a) one may integrate over s_2 *via* (2.4):

$$\theta(s_{23}-\omega)\int d\phi_2\,\xi_2\,s_{23}^{\alpha_{c,2}-\alpha_{c,1}-\alpha_{c,3}-2}\int_{\gamma s_{23}/\omega}^{s_{23}} ds_3\,\frac{\pi}{\Gamma(\alpha(t)+1)}\,g(t;t_1^{\pm})\,g(t;t_2^{\pm})$$

$$\times\left[\frac{s_{23}\gamma}{s_3}\right]^{\alpha(t)-\alpha_{c,2}}\frac{1}{\alpha(t)-\alpha_{c,2}}\,(s_3^{\perp})^{\alpha_{c,3}+1}\,\frac{1}{2i}\,\Delta A_{R_2R_3}(s_3,s_3^{\perp};t;t_2^{\pm},t_3^{\pm})\,. \quad (3.5)$$

Now for sufficiently large s_3,

$$s_3^{\perp}\approx s_3\,, \quad (3.6)$$

and the high-energy form (2.3) of $(1/2i)\,\Delta A_{R_2R_3}(s_3,\ldots)$ may be used, so that (3.5) becomes

$$\text{(I)}\quad \theta(s_{23}-\omega)\,\frac{\pi g(t;t_1^{\pm})\,g(t;t_3^{\pm})}{\Gamma(\alpha(t)+1)}\,Q(t)\ln\left(\frac{\omega}{\gamma}\right)s_{23}^{\alpha(t)-\alpha_{c,1}-\alpha_{c,3}-2}\,, \quad (3.7)$$

where

$$Q(t)\equiv\frac{\pi}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\xi_2\,g^2(t;t_2^{\pm})\,\frac{\gamma^{\alpha(t)-\alpha_{c,2}}}{\alpha(t)-\alpha_{c,2}}\,. \quad (3.8)$$

Similarly one finds the contribution of (3.4b) to be

$$\text{(II)}\quad \theta(s_{23}-\omega)\,\frac{\pi g(t;t_1^{\pm})\,g(t;t_3^{\pm})}{\Gamma(\alpha(t)+1)}\,Q(t)\ln\left(\frac{s_{23}}{\omega}\right)s_{23}^{\alpha(t)-\alpha_{c,1}-\alpha_{c,3}-2}\,, \quad (3.9)$$

while the contribution of (3.4c) is

$$\text{(III)}\quad \theta(s_{23}-\omega)\,\frac{\pi g(t;t_1^{\pm})\,g(t;t_3^{\pm})}{\Gamma(\alpha(t)+1)}\,V(t)\,s_{23}^{\alpha(t)-\alpha_{c,1}-\alpha_{c,3}-2}\,, \quad (3.10)$$

where

$$V(t)\equiv\frac{\pi}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\xi_2\,g^2(t;t_2^{\pm})\,\frac{\gamma^{\alpha(t)-\alpha_{c,2}}}{[\alpha(t)-\alpha_{c,2}]^2}\,. \quad (3.11)$$

Thus the one-twist term is, for $s_{23} > \omega$,

$$C^{(1)}_{R_1R_3}(s_{23}, t; t_1^\pm, t_3^\pm) = \theta(s_{23} - \omega)[Q(t)\ln(s_{23}/\gamma) + V(t)]$$

$$\times s_{23}^{\alpha(t)-\alpha_{c,1}-\alpha_{c,3}-2} \frac{\pi}{\Gamma(\alpha(t)+1)} g(t; t_1^\pm) g(t; t_3^\pm) . \tag{3.12}$$

From the planar bootstrap condition (2.6) one may estimate $Q(t)$, $V(t) \sim O(1/N)$ so that the order of (3.12) is consistent with TE expectations.

To transform (3.12) into the j-plane, one may introduce the Mellin transform

$$\tilde{C}^{(1)}_{R_1R_3}(J) \equiv \int_\omega^\infty ds_{23} s_{23}^{-J-1} C^{(1)}_{R_1R_3}(s_{23}; t; t_1^\pm, t_3^\pm) , \tag{3.13}$$

so that

$$\tilde{C}^{(1)}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2)$$

$$= \frac{\pi g(t; t_1^\pm) g(t; t_3^\pm)\, \omega^{\alpha(t)-j}}{\Gamma(\alpha(t)+1)[j-\alpha(t)]} \left\{ \frac{Q(t)}{[j-\alpha(t)]} + V(t) + Q(t) \ln\frac{\omega}{\gamma} \right\} \tag{3.14}$$

$$= \frac{\pi g(t; t_1^\pm) g(t; t_3^\pm)\, \omega^{\alpha(t)-j}}{\Gamma(\alpha(t)+1)[j-\alpha(t)]} \left\{ \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \xi_2 \frac{g^2(t; t_2^\pm)}{\alpha(t) - \alpha_{c,2}} \gamma^{\alpha(t)-\alpha_{c,2}} \right.$$

$$\left. \times \left[\ln\frac{\omega}{\gamma} + \frac{j - \alpha_{c,2}}{[\alpha(t) - \alpha_{c,2}]} \frac{1}{[j - \alpha(t)]} \right] \right\} . \tag{3.15}$$

This simple result is *not* what one gets if analyticity is neglected and only the high-energy form of planar amplitudes is used. To see this, consider the widely used high-energy approximation (HE):

$$\int_0^\infty ds_{23} s_{23}^{-j+\alpha_{c,1}+\alpha_{c,3}+2-1} \frac{1}{2i} \Delta A_{R_1R_3}(s_{23}, s_{23}^\perp; t; t_1^\pm, t_3^\pm)$$

$$\approx \frac{\pi g(t; t_1^\pm) g(t; t_3^\pm)}{\Gamma(\alpha(t)+1)[j-\alpha(t)]} \equiv \frac{1}{2i} \Delta \tilde{A}^{HE}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2) , \tag{3.16}$$

$$s_i^\perp \approx s_i . \tag{3.17}$$

Apart from having no *a priori* justification, the HE approximation generates j-plane planar unitarity equations (of the type studied by Kwiecinski and Sakai [18]) which are incompatible with a pure pole bootstrap. Nevertheless, consider the one-twist term in this HE approximation. It is usually written

$$\tilde{C}^{(1)\mathrm{HE}}_{\mathrm{R_1R_3}}(j-\alpha_{c,1}-\alpha_{c,3}-2)=\int \mathrm{d}\phi_2\xi_2\,\frac{1}{2i}\,\Delta\tilde{A}^{\mathrm{HE}}_{\mathrm{R_1R_2}}(j-\alpha_{c,1}-\alpha_{c,2}-2)$$

$$\times\frac{\gamma^{j-\alpha_{c,2}}}{j-\alpha_{c,2}}\,\frac{1}{2i}\,\Delta\tilde{A}^{\mathrm{HE}}_{\mathrm{R_2R_3}}(j-\alpha_{c,2}-\alpha_{c,3}-2) \tag{3.18}$$

$$=\frac{\pi g(t;t_1^{\ddagger})\,g(t;t_3^{\ddagger})}{\Gamma(\alpha(t)+1)}\int \mathrm{d}\phi_2\xi_2\,\frac{\gamma^{j-\alpha_{c,2}}\,g^2(t;t_2^{\ddagger})}{[j-\alpha_{c,2}]\,[j-\alpha(t)]^2}\,\frac{\pi}{\Gamma(\alpha(t)+1)}\,, \tag{3.19}$$

which fails to reproduce the form of the precise result (3.15). This failure is not surprising since already in strictly planar calculations it is misleading to ignore vital analyticity properties such as the FMSR (2.4) and (2.5) (see also subsect. 3.2.2 and appendix B of ref. [10]).

The important point here is that, contrary to widespread practice, the Mellin transform does *not* in general diagonalize the one-twist term into a product of symmetric transforms over the planar blobs and the usual $(j-\alpha_{c,2})^{-1}$ internal reggeon loop propagator.

4. Cylinder constructions

4.1. A cylinder construction maintaining planar analyticity

Let me now turn to the cylinder integration equation of fig. 4. Denoting the cylinder by $C_{\mathrm{R_1R_3}}(s_{23};t;t_1^{\ddagger},t_3^{\ddagger})$, fig. 4 suggests:

$$C_{\mathrm{R_1R_3}}(s_{23};t;t_1^{\ddagger},t_3^{\ddagger})=C^{(1)}_{\mathrm{R_1R_3}}(s_{23};t;t_1^{\ddagger},t_3^{\ddagger})+N\int \mathrm{d}\phi_2\xi_2 s_{23}^{\alpha_{c,2}-\alpha_{c,1}-\alpha_{c,3}-2}$$

$$\times\theta(s_{23}-\omega)\int_0^{s_{23}}\mathrm{d}s_2\int_0^{s_{23}}\mathrm{d}s_3\,\theta(\gamma s_{23}-s_2s_3)(s_2^{\perp})^{\alpha_{c,2}+1} \tag{4.1}$$

$$\times\frac{1}{2i}\,\Delta A_{\mathrm{R_1R_2}}(s_2,s_2^{\perp};t;t_1^{\ddagger},t_2^{\ddagger})(s_3^{\perp})^{\alpha_{c,3}+1}\,C_{\mathrm{R_2R_3}}(s_3;t;t_2^{\ddagger},t_3^{\ddagger})\,\theta(s_3-\omega)\,,$$

where ω is now identified as the cylinder threshold. The threshold ω is so chosen that it exceeds the planar threshold and is sufficiently large so that for $s_3>\omega$, one may take $s_3\approx s_3^{\perp}$. Note that the N factor in (4.1) is required by the closed internal quark loops in the 2,3,... twist graphs (see fig. 3). Rewriting (4.1),

$$C_{\mathrm{R_1R_3}}(s_{23};t;t_1^{\ddagger},t_3^{\ddagger})=C^{(1)}_{\mathrm{R_1R_3}}(s_{23};t;t_1^{\ddagger},t_3^{\ddagger})+N\int \mathrm{d}\phi_2\xi_2 s_{23}^{\alpha_{c,2}-\alpha_{c,1}-\alpha_{c,3}-2}$$

$$\times\theta(s_{23}-\omega)\int_{\omega}^{s_{23}}\mathrm{d}s_3\int_0^{\gamma s_{23}/s_3}\mathrm{d}s_2\,(s_2^{\perp})^{\alpha_{c,1}+1}$$

$$\times \frac{1}{2i} \Delta A_{R_1R_2}(s_2, s_{\bar{2}}; t; t_1^{\pm}, t_3^{\pm})\, s_3^{\alpha_{c,3}+1}\, C_{R_2R_3}(s_3; t; t_2^{\pm}, t_3^{\pm}) , \tag{4.2}$$

one may readily perform the s_2 integration *via* FMSR (2.4):

$$C_{R_1R_3}(s_{23}; t; t_1^{\pm}, t_3^{\pm}) = C^{(1)}_{R_1R_3}(s_{23}; t; t_1^{\pm}, t_3^{\pm}) + N \int d\phi_2 \xi_2 s_{23}^{\alpha_{c,2}-\alpha_{c,1}-\alpha_{c,3}-2}$$

$$\times \theta(s_{23} - \omega) \int_{\omega}^{s_{23}} ds_3 \frac{\pi g(t; t_1^{\pm})\, g(t; t_3^{\pm})}{\Gamma(\alpha(t)+1)[\alpha(t) - \alpha_{c,2}]}$$

$$\times \left[\frac{s_{23}\gamma}{s_3}\right]^{\alpha(t)-\alpha_{c,2}} s_3^{\alpha_{c,3}+1}\, C_{R_2R_3}(s_3; t; t_2^{\pm}, t_3^{\pm}) . \tag{4.3}$$

The Mellin transform

$$\widetilde{C}_{R_1R_3}(J) \equiv \int_{\omega}^{\infty} ds_{23} s_{23}^{-J-1}\, C_{R_1R_3}(s_{23}; t; t_1^{\pm}, t_3^{\pm}) \tag{4.4}$$

projects (4.3) into the j-plane equation:

$$\widetilde{C}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2) = \widetilde{C}^{(1)}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2)$$

$$+ \frac{\pi g(t; t_1^{\pm})\, N}{\Gamma(\alpha(t)+1)[j - \alpha(t)]} \int d\phi_2 \xi_2 \frac{g(t; t_2^{\pm})}{\alpha(t) - \alpha_{c,2}}$$

$$\times \widetilde{C}_{R_2R_3}(j - \alpha_{c,2} - \alpha_{c,3} - 2)\, \gamma^{\alpha(t)-\alpha_{c,2}} . \tag{4.5}$$

This integral equation is easily solved:

$$\widetilde{C}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2) = \widetilde{C}^{(1)}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2)$$

$$+ \frac{\dfrac{\pi g(t; t_1^{\pm})\, N}{\Gamma(\alpha(t)+1)[j - \alpha(t)]} \displaystyle\int d\phi_2 \xi_2 \frac{g(t; t_2^{\pm})\, \gamma^{\alpha(t)-\alpha_{c,2}}}{\alpha(t) - \alpha_{c,2}}\, \widetilde{C}^{(1)}_{R_2R_3}(j - \alpha_{c,2} - \alpha_{c,3} - 2)}{1 - \dfrac{N\pi}{[j - \alpha(t)]\, \Gamma(\alpha(t)+1)} \displaystyle\int d\phi_2 \xi_2 g^2(t; t_2^{\pm}) \frac{\gamma^{\alpha(t)-\alpha_{c,2}}}{\alpha(t) - \alpha_{c,2}}} . \tag{4.6}$$

Substituting the one-twist term (3.15) into (4.6) yields

$$\widetilde{C}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2) = \frac{\widetilde{C}^{(1)}_{R_1R_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2)}{1 - NQ(t)/[j - \alpha(t)]} \tag{4.7}$$

$$= \frac{\pi g(t; t_1^{\pm})\, g(t; t_3^{\pm})\, \omega^{\alpha(t)-j}}{\Gamma(\alpha(t)+1)\, N} \left\{\frac{NV(t) + 1 + NQ(t) \ln(\omega/\gamma)}{j - \alpha(t) - NQ(t)} - \frac{1}{j - \alpha(t)}\right\} , \tag{4.8}$$

or equivalently, in the energy plane

$$C_{R_1R_3}(s_{23}; t; t_1^\pm, t_3^\pm) = \frac{\pi g(t; t_1^\pm)\, g(t; t_3^\pm)}{\Gamma(\alpha(t)+1)\, N}\, \theta(s_{23} - \omega)$$

$$\times \{\omega^{-Q(t)}[NV(t) + 1 + NQ(t)\ln(\omega/\gamma)]$$

$$\times s_{23}^{\alpha(t)+NQ(t)-\alpha_{c,1}-\alpha_{c,3}-2} - s_{23}^{\alpha(t)-\alpha_{c,1}-\alpha_{c,3}-2}\} . \tag{4.9}$$

This cylinder displays a leading j-plane pole, the bare pomeron singularity P with a trajectory function $\alpha_P(t)$, where

$$\alpha_P(t) = \alpha(t) + \frac{\pi N}{\Gamma(\alpha(t)+1)} \int d\phi_2\, \xi_2\, \frac{g^2(t; t_2^\pm)\, \gamma^{\alpha(t)-\alpha_{c,2}}}{\alpha(t) - \alpha_{c,2}}$$

$$= \alpha(t) + NQ(t) , \tag{4.10}$$

which is completely determined by quantities already appearing on the planar level. To get a qualitative estimate of the bare pomeron parameters, we use the coupling normalization condition (2.6). Going to $t = 0$, one has:

$$1 = \frac{\pi N}{\Gamma(\alpha(0)+1)} \int dt_2^+ \int dt_2^- \frac{\delta(t_2^+ - t_2^-)\, \gamma^{\alpha(0)-\alpha_{c,2}}\, g^2(0; t_2^\pm)}{16\pi^3[\alpha(0) - \alpha_{c,2}]^2}$$

$$\times \Gamma(-\alpha(t_2^+))\, \Gamma(-\alpha(t_2^-)) . \tag{4.11}$$

The bare pomeron intercept now becomes

$$\alpha_P(0) = \alpha(0) + \frac{\pi N}{\Gamma(\alpha(0)+1)} \int dt^+ \int dt^- \frac{\delta(t_2^+ - t_2^-)\, \gamma^{\alpha(0)-\alpha_{c,2}}\, g^2(0; t_2^\pm)}{16\pi^3[\alpha(0) - \alpha_{c,2}]}$$

$$\times \Gamma(-\alpha(t_2^+))\, \Gamma(-\alpha(t_2)) , \tag{4.12}$$

$$= \alpha(0) + \langle \alpha(0) - \alpha_c \rangle_{\text{average}}$$

$$= 1 + 2\alpha(0) - 2\langle \alpha(t^+) \rangle_{\text{average}} . \tag{4.13}$$

For approximately linear Regge trajectories

$$\langle \alpha(t^+) \rangle_{\text{average}} \approx \alpha(0) + \alpha' \langle t^+ \rangle_{\text{average}} , \tag{4.14}$$

where $\alpha' \approx 1\ \mathrm{GeV}^{-1}$ is the Regge slope. Thus

$$\alpha_{\mathrm{P}}(0) \approx 1 - 2\alpha' \langle t^+ \rangle_{\mathrm{average}} . \tag{4.15}$$

The bare pomeron intercept is slightly above unity!

The residue strength of the bare pomeron may similarly be estimated. Taking $\omega \approx \gamma \approx 1$ GeV one finds that at $t = 0$ this residue is $\sim 2N^{-1} g(0; t_1^{\pm}) g(0; t_3^{\pm})\, \pi / \Gamma(\alpha(0) + 1)$.

The bare pomeron slope is rather sensitive to the choice of triple-Regge couplings and will be analyzed in a future work.

In addition to the leading, bare pomeron pole, the cylinder (4.8) has another pole at $j = \alpha(t)$ with negative residue of the same strength as the SU(N) singlet component of the planar pole. Thus, at sufficiently high energies, where planar amplitudes may be approximated by their Regge asymptotic forms, the cylinder (4.8) in effect extinguishes the planar singlet:

$$C_{\mathrm{R}_1\mathrm{R}_3}(s_{23}; t; t_1^{\pm}, t_3^{\pm}) + \frac{1}{2i} \Delta A^{\mathrm{sing}}_{\mathrm{R}_1\mathrm{R}_3}(s_{23}, s_{23}^{\perp}; t; t_1^{\pm}, t_3^{\pm})$$

$$\xrightarrow[\text{small } t]{\text{large } s_{23}} \sim \frac{N^{-1} \pi 2 g(t; t_1^{\pm})\, g(t; t_3^{\pm})}{\Gamma(\alpha(t) + 1)}\, s_{23}^{\alpha_{\mathrm{P}}(t) - \alpha_{c,1} - \alpha_{c,3} - 2} . \tag{4.16}$$

Such a mechanism, whereby the bare pomeron may be viewed simply as a shifted planar, vacuum-quantum-number trajectory, has been advocated in several papers [4,5,19].

In the one-dimensional limit where transverse variables (i.e. momentum transfers) are suppressed, the bare pomeron intercept (4.15) becomes exactly one. In fact the cylinder constructed in this section is the three-dimensional generalization of the simple, one-dimensional Huan Lee model [12] that was based on diagram counting.

4.2. An approximate cylinder construction neglecting planar analyticity

It is worthwhile to contrast cylinder (4.8) with the usual constructions that neglect planar analyticity. With the high-energy approximation (3.16) and (3.17) one writes the approximate j-plane cylinder equation:

$$\widetilde{C}^{\mathrm{HE}}_{\mathrm{R}_1\mathrm{R}_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2) = \widetilde{C}^{(1)\mathrm{HE}}_{\mathrm{R}_1\mathrm{R}_3}(j - \alpha_{c,1} - \alpha_{c,3} - 2)$$

$$+ N \int \mathrm{d}\phi_2\, \xi_2\, \frac{1}{2i}\, \Delta \widetilde{A}^{\mathrm{HE}}_{\mathrm{R}_1\mathrm{R}_2}(j - \alpha_{c,1} - \alpha_{c,2} - 2)\, \frac{\gamma^{j - \alpha_{c,2}}}{j - \alpha_{c,2}}$$

$$\times\, \widetilde{C}^{\mathrm{HE}}_{\mathrm{R}_2\mathrm{R}_3}(j - \alpha_{c,2} - \alpha_{c,3} - 2) . \tag{4.17}$$

Substituting (3.16) and (3.19) into (4.17), one finds the solution

$$\tilde{C}^{\mathrm{HE}}_{\mathrm{R_1R_3}}(j - \alpha_{c,1} - \alpha_{c,3} - 2) = \frac{\tilde{C}^{(1)\mathrm{HE}}_{\mathrm{R_1R_3}}(j - \alpha_{c,1} - \alpha_{c,3} - 2)}{1 - \dfrac{\pi N}{\Gamma(\alpha(t)+1)} \displaystyle\int \frac{\mathrm{d}\phi_2\, \xi_2\, g^2(t; t_2^{\pm})\, \gamma^{j-\alpha_{c,2}}}{[j - \alpha(t)]\,[j - \alpha_{c,2}]}} \tag{4.18}$$

$$= \frac{\pi g(t; t_1^{\pm})\, g(t; t_3^{\pm})}{\Gamma(\alpha(t)+1)\, N} \left\{ \frac{1}{j - \alpha(t) - \dfrac{\pi N}{\Gamma(\alpha(t)+1)} \displaystyle\int \mathrm{d}\phi_2\, \xi_2\, \frac{g^2(t; t_2^{\pm})\, \gamma^{j-\alpha_{c,2}}}{j - \alpha_{c,2}}} - \frac{1}{j - \alpha(t)} \right\}, \tag{4.19}$$

which is essentially equivalent to Bishari's cylinder [6]. Instead of a simple pole singularity structure as in (4.8) the approximate cylinder in (4.19) is contaminated by j-plane cuts. The singularity structure of (4.19) is now determined by a complicated eigenvalue equation [6]. Furthermore, the oversimplified planar reggeon amplitudes (3.16) used to build the cylinder (4.19) are not themselves compatible with a self-consistent pole-to-pole planar unitarity; so one wonders in what sense the triple-Regge couplings in (4.19) are to be normalized. This last point is well-illustrated by taking the one-dimensional limit (and setting $\gamma \approx 1$). The leading pole of (4.19) is now determined from

$$[j - \alpha]\,[j - \alpha_c] = \frac{N\pi}{\Gamma(\alpha + 1)} g^2 , \tag{4.20}$$

where now $\alpha_c = 2\alpha - 1$. A unit intercept bare pomeron is now forbidden if $\alpha \approx \frac{1}{2}$. (See also sect. 3 of ref. [10].) Since this model certainly does not correspond to the usual diagram-counting model of Huan Lee [12] it is now dubious to adopt the planar bootstrap normalization condition:

$$1 = \frac{N\pi g^2}{\Gamma(\alpha + 1)(\alpha - \alpha_c)^2} , \tag{4.21}$$

which *can* be derived by diagram counting.

5. Remarks and conclusions

(i) If planar reggeon amplitudes consistent with a cut-free, pole-to-pole planar bootstrap are sewn together to generate a (exchanged) cylinder then this cylinder will itself be free of j-plane cuts. Interestingly, no further information about reggeon amplitudes other than that already needed for planar self-consistency (namely, Regge asymptotic behavior and certain good FMSR) is used in calculating the cylinder of subsect. 4.1. This cylinder has a leading singularity, the bare pomeron pole, with intercept slightly above one. In addition, this cylinder has a negative residue pole that exactly cancels with the singlet component of the high-energy planar exchange.

(ii) If, on the other hand, planar reggeon amplitudes are approximated by only their high-energy form then the corresponding approximate cylinder of subsect. 4.2 has a rather complicated singularity structure contaminated by j-plane cuts.

(iii) The intriguing question whether a cylinder model can be formulated that avoids pomeron-f identity but still has a simple j-plane singularity structure warrants investigation. For example, the simple (but mysterious) dynamical assumption that in building up the cylinder, the leading particles (at the edges of the chain) must be emitted from different quark lines alters the cylinder equation (4.1) so as to generate only the bare pomeron pole singularity (4.10) with no f-extinction piece.

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