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Cancellation of unconventional terms in the planar bootstrap

J. R. Freeman and C. E. Jones
Behlen Laboratory of Physics, University of Nebraska, Lincoln, Nebraska 68588
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Previous work has shown that in order for a planar pole bootstrap to exist without Regge cuts in the topological expansion the physical discontinuity across certain Reggeon amplitudes must have unconventional terms in its asymptotic behavior. Here we investigate whether such unconventional asymptotic terms enter self-consistently into the planar unitarity equation. We find, in fact, that such terms are actually needed in the unitarity equation to cancel heretofore overlooked contributions to unitarity.

I. INTRODUCTION

We examine here certain self-consistency questions concerning the planar approximation to the topological expansion[1] (TE) of the strong-interaction S matrix. In this approximation the so-called "ordered" S matrix is assumed to be exactly unitary. (The ordered S matrix[2] is defined to act on an enlarged unphysical Hilbert space where different orderings of the particle labels correspond to different states in the Hilbert space.) The self-consistent unitary ordered S matrix is then the basic object in terms of which higher-order corrections to the S matrix are made, these corrections corresponding to more complicated, nonplanar topologies.

A central problem then of the TE program is to determine the ordered S matrix through the self-consistency conditions of unitarity. It has been conjectured that the ordered S matrix (and, hence, the planar approximation, which consists of sums over the ordered S matrix elements) will have only Regge poles and no Regge cuts.

Recent work by the present authors[3,4] has shown on general grounds that, if a pure pole bootstrap does exist for the ordered S matrix, certain unconventional asymptotic terms must be present in Reggeon amplitudes to ensure Regge-cut cancellations in unitarity. At first sight these unconventional terms seem rather unnatural (although they do serve the purpose of keeping planar amplitudes finite[4] when the leading Regge trajectory passes through −1). Whereas these unconventional terms do guarantee Regge-cut cancellation, the question arises whether these terms generate through unitarity new output pole singularities that violate a self-consistent Regge-pole bootstrap.

In this paper we study in some detail how these unconventional asymptotic terms manifest themselves in a simple form of the unitary self-consistency equations for the ordered amplitudes. This study suggests that such terms, far from being awkward, are actually essential in making the unitary equations sensible. This result is somewhat remarkable in view of the fact that the original arguments requiring the presence of such unconventional terms were based primarily on analyticity and not on unitarity.

II. SELF-CONSISTENT UNITARITY WITH REGGE POLES

The basic unitarity equation we consider is for the four-line connected (ordered) amplitude with two external Reggeons and it is depicted in Fig. 1. In this unitarity relation all amplitudes on both sides of the equation in Fig. 1 are ordered. This means, in particular, that no twists occur in the particle lines for the n-particle intermediate states on the right side of Fig. 1. The simpler problem of unitarity for the four-line particle amplitude ($\alpha_1 = 0$) presents no problems with respect to the unconventional asymptotic terms and so we consider it here only as a limiting case.

To implement the writing of unitarity in a way which will permit a precise treatment of phase space we (i) work in the forward ($t=0$) direction; (ii) separate the n-particle ordered intermediate state in Fig. 1 into two "clusters," the left cluster consisting of a single particle (the first particle in the order) and the right cluster consisting of the remaining particles; (iii) insert Regge-pole exchanges between the first and second particles in

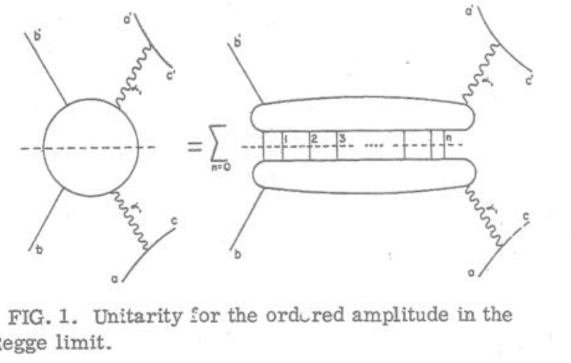

FIG. 1. Unitarity for the ordered amplitude in the Regge limit.

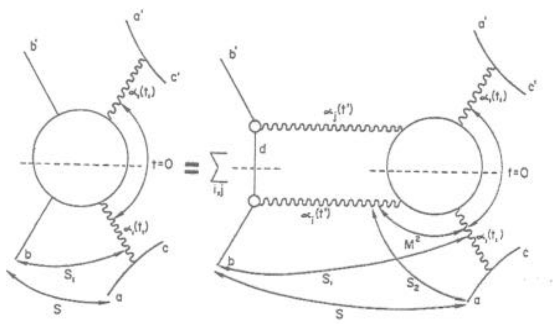

FIG. 2. Unitarity relation in terms of the Reggeon-Reggeon total cross section.

the intermediate state. The result of these steps is depicted in Fig. 2.

The assumptions going into the unitarity relation of Fig. 2 are that only Regge-pole exchanges need be introduced (no Regge cuts) and that by summing over sufficiently many α_i the $2\to n$ amplitude is well-enough represented even in phase-space regions where the subenergy of the first two particles in the intermediate state may not be large. These assumptions will have to be scrutinized later when we shall argue that, in fact, the approximation to unitarity in Fig. 2 does leave out an important contribution which relates to the unconventional asymptotic terms.

The advantage of the treatment of unitarity according to Fig. 2 is that due to factorization of Regge poles, the unitarity sum is determined entirely in terms of Regge-particle residues and the appropriate physical discontinuity of a Reggeon-Reggeon amplitude. (To see how Reggeon amplitudes are extracted from appropriate particle amplitudes see Ref. 5.)

The unitarity equation representing Fig. 2 can be written (for large $s\to\infty$) as follows:

$$(-s)^{2\alpha_1(t_1)}\ \frac{1}{2i}\ \Delta_{s_1}A_{b,\alpha_1}(s_1;t=0;t_1)$$

$$=(-s)^{2\alpha_1(t_1)}\sum_{i,j}N\int^0\frac{dt'}{16\pi^3}\,(s_1)^{\alpha_i(t')+\alpha_j(t')-2\alpha_1(t_1)-1}\beta_{bd}^{\alpha_i}(t')$$

$$\times\beta_{bd}^{\alpha_j}(t')\Gamma(-\alpha_i(t'))\,\Gamma(-\alpha_j(t'))\cos\pi[\alpha_i(t')-\alpha_j(t')]$$

$$\times\int_{M_0^2}^{s_1f(t')}dM^2\left[\frac{1}{2i}\ \Delta_{M^2}A_{\alpha_i,\alpha_1}(M^2,M_\perp{}^2;t=0;t',t_1)\right](M_\perp{}^2)^{2\alpha_1(t_1)}\ ,\tag{2.1}$$

where common residue factors have been canceled; the β's are particle-particle-Reggeon couplings; $M_\perp{}^2\equiv s_1s_2/s$ is linearly related to M^2; N comes from the assumed internal SU(N) symmetry; $f(t')$ is determined by phase-space considerations and is less than one; A_{b,α_1} and A_{α_j,α_1} are particle-Reggeon and Reggeon-Reggeon amplitudes, respectively.

In the large-s_1 limit of Fig. 2, the leading Regge pole in the $a\bar{a}$ channel controls the asymptotic behavior of the left-hand side of the unitarity equation. In particular

$$\frac{1}{2i}\ \Delta_{s_1}A_{b,\alpha_1}\underset{s_1\to\infty}{\sim}\ \beta_{bb}^{\alpha}(0)g_{\alpha_1,\alpha_1}^{\alpha}(t=0;t_1)$$

$$\times\frac{\pi s_1{}^{\alpha(0)-2\alpha_1(t_1)}}{\Gamma(\alpha(0)+1)}+\cdots,\tag{2.2}$$

where g is the triple-Regge coupling and $\alpha(0)$ is the Regge pole exchanged in the t channel. Subdominant terms in (2.2) will consist of a sum over powers $\alpha(0)$ consistent with the assumption of only Regge-pole exchange in planar amplitudes.

An important question is whether such pure pole behavior can be consistent with the right-hand side of (2.1) or Fig. 2. The integral over t' in (2.1) appears to imply Regge-cut behavior at large s_1. This cut behavior can only be averted if the integral over M^2 in the unitarity sum (2.1) has the form

$$\int_{M_0^2}^{s_1f(t')}dM^2(M_\perp{}^2)^{2\alpha_1(t_1)}$$

$$\times\ \frac{1}{2i}\ \Delta_{M^2}A_{\alpha_i,\alpha_1}(M^2,M_\perp{}^2;t=0;t',t_1)$$

$$=\sum_\omega D_{ij}(\omega,t',t_1)s_1{}^{\omega-\alpha_i(t')-\alpha_j(t')+1}.\tag{2.3}$$

Note, in particular, that no constant terms independent of s_1 can appear in (2.3)—such as might be expected from the lower limit of integration—or else cut behavior in (2.1) is inevitable.[3-5] In the case (2.3) is obeyed, the integrand in (2.3) is said to satisfy a "good" finite-mass sum rule (FMSR).

In order to proceed we shall assume that the four-Reggeon amplitude A_{α_i,α_1} has properties that are generalizations of the two-Reggeon amplitude discussed in Refs. 3 and 4. In the latter case, in order for a good FMSR to be satisfied, an unconventional term in the asymptotic behavior was required (whose presence was dictated by the fact that the two-Reggeon amplitude possessed a left-hand cut in M^2). In the case of A_{α_i,α_1}, there is also a left-hand cut in M^2 (at $M_\perp{}^2=0$) which leads to an unconventional asymptotic term in (2.3) with $\omega = -1$. Thus we have the following asymptotic behavior for A_{α_i,α_1}:

$$\frac{1}{2i}\Delta_{M^2}A_{\alpha_i,\alpha_1} \underset{M^2\to\infty}{\sim} g^{\alpha}_{\alpha_i,\alpha_j}(t=0;t')g^{\alpha}_{\alpha_1,\alpha_1}(t=0;t_1)\left[\frac{\pi}{\Gamma(\alpha(0)+1)}(M^2)^{\alpha(0)-2\alpha_1(t_1)-\alpha_i(t')-\alpha_j(t')}\right] + G(t=0;t',t_1)(M^2)^{-1-2\alpha_1(t_1)-\alpha_i(t')-\alpha_j(t')}+\cdots, \tag{2.4}$$

where the additional terms are down by integer powers of M^2. The FMSR (2.3) has the form

$$\int_{M_0{}^2}^{s_1 f(t')} dM^2(M_\perp{}^2)^{2\alpha_1(t_1)}\frac{1}{2i}\Delta_{M^2}A_{\alpha_i,\alpha_1}(M^2,M_\perp{}^2;t=0;t',t_1) = \frac{\pi}{\Gamma(\alpha(0)+1)}g^{\alpha}_{\alpha_i,\alpha_j}(t=0;t')g^{\alpha}_{\alpha_1,\alpha_1}(t=0;t_1)\frac{[s_1 f(t')]^{\alpha(0)-\alpha_i(t')-\alpha_j(t')+1}}{\alpha(0)-\alpha_i(t')-\alpha_j(t')+1} + \overline{G}(t=0;t',t_1)[s_1 f(t')]^{-\alpha_i(t')-\alpha_j(t')}+\cdots. \tag{2.5}$$

We point out, as already emphasized in Ref. 3, the dual-resonance-model Reggeon amplitudes will *not* satisfy this good FMSR needed for cut cancelation.

Inserting FMSR (2.5) into the unitarity equation (2.1), we then match up powers of s_1 on both sides of the equation. First, matching the leading Regge powers, we find the following self-consistency condition[6] on the residue functions:

$$\beta^{\alpha}_{bb'}(0)=\sum_{i,j} N\int^0 \frac{dt'}{16\pi^3}\beta^{\alpha_i}_{bd}(t')\beta^{\alpha_j}_{b'd}(t')\Gamma(-\alpha_i(t'))\Gamma(-\alpha_j(t'))\frac{\pi}{\Gamma(\alpha(0)+1)} \times\frac{\cos\pi[\alpha_i(t')-\alpha_j(t')]g^{\alpha}_{\alpha_i,\alpha_j}(t=0;t')}{\alpha(0)-\alpha_i(t')-\alpha_j(t')+1}[f(t')]^{\alpha(0)-\alpha_i(t')-\alpha_j(t')+1}. \tag{2.6}$$

However, (2.6) does not give complete self-consistency since the unconventional terms in (2.4) and (2.5) will generate a term on the right-hand side of (2.1), with the behavior

$$(-s)^{2\alpha_1(t_1)}(s_1)^{-1-2\alpha_1(t_1)}.$$

Such a term has *no* counterpart on the left-hand side of the unitarity equation. There are no unconventional terms appearing in the asymptotic form (2.2) of $(1/2i)\Delta A_{b,\alpha_1}$ since this amplitude has no left-hand cut in energy. We now wish to argue that a term of the form

$$(-s)^{2\alpha_1(t_1)}(s_1)^{-1-2\alpha_1(t_1)}$$

is actually *needed* in order to cancel a similar term that has so far been neglected on the right-hand side of (2.1).

A scrutiny of the unitarity sum shown in Fig. 2 reveals that certain s_1 discontinuities, shown symbolically by the Feynman graph shown in Fig. 3, have not yet been included on the right-hand side of Fig. 2 or Eq. (2.1). The graph in Fig. 3 is of the type studied by Mandelstam[7] in connection with cut cancellation mechanisms. Its asymptotic behavior is given by (see Ref. 7)

$$\frac{1}{2i}\Delta_{s_1}A_{\text{cut}} \underset{s,s_1\to\infty}{\sim} (-s)^{2\alpha_1(t_1)}(s_1)^{-1-2\alpha_1(t_1)} \times \text{residues}. \tag{2.7}$$

Thus (2.7) must be added to the right-hand side of (2.1). This contribution is of exactly the same form as that appearing as a result of the unconventional asymptotic term in A_{α_i,α_1}. These two terms must cancel in order that such behavior be absent from $(1/2i)\Delta A_{b,\alpha_1}$. Thus we see that the unconventional asymptotic term in A_{α_i,α_1} actually *must* be present in order to cancel the other unwanted term.

It is important to check that our argument for the

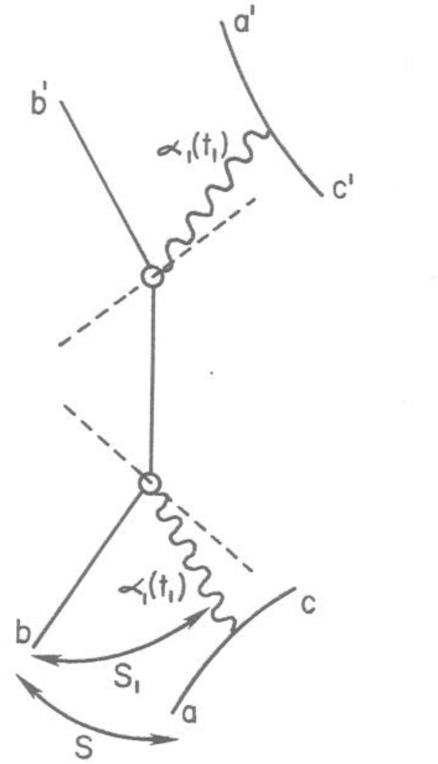

FIG. 3. Graphs omitted from the unitarity sum of Fig. 2.

cancellation of the unwanted term does not run into contradictions when we look in the limit $\alpha_1 \to 0$ where the unitarity equations refer to the four-line particle amplitude. In this limit Fig. 2 becomes Fig. 4 and Fig. 3 becomes Fig. 5. Also, the amplitude A_{α_i,α_1} is replaced by $A_{\alpha_i,a}$ (see Fig. 4).

It is clear that $A_{\alpha_i,a}$ has the same general asymptotic properties as A_{b,α_i} shown in Eq. (2.2). In particular $(1/2i)\Delta A_{\alpha_i,a}$ has no unconventional terms in its asymptotic behavior and thus the unitarity sum indicated by Fig. 4 will have no such terms. But by the same token the graph of Fig. 5 has no discontinuity above threshold and thus has no need of a canceling term. Thus there is

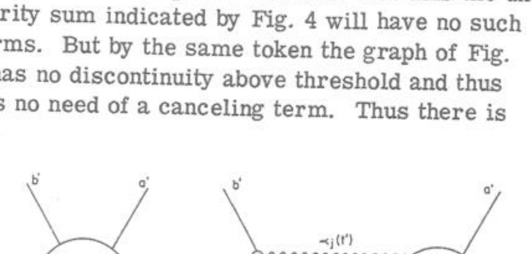

FIG. 4. Unitarity for the four-line particle amplitude.

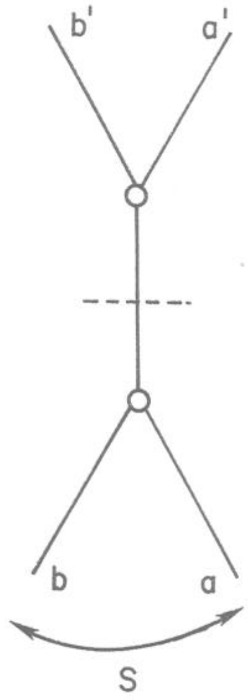

FIG. 5. Graph omitted from the unitarity sum of Fig. 4.

no problem of unconventional terms in the limit $\alpha_1 \to 0$.

III. DISCUSSION

The use of Feynman diagrams (Fig. 3) in a paper dealing with S-matrix theory and equations of unitarity perhaps deserves some discussion. One does not expect to need the use of Feynman diagrams in S-matrix theory. However, in the present case, the need arose because our procedure for performing the unitarity summation was carried out in such a way as to explicitly exclude certain contributions that can best be described by the Feynman diagram of Fig. 3.

To clarify this point further, we indicate the manner in which exact unitarity, as opposed to the approximate form given in Fig. 2 with Regge exchanges $\alpha_i(t')$ and $\alpha_j(t')$, will in fact automatically include the contributions of Fig. 3. The exact unitarity without assuming Regge exchanges is shown in Fig. 1. We see that by letting the lower blob on the right side of Fig. 1 be a one-particle amplitude as shown in Fig. 6 we pickup the unitarity cuts of Fig. 3 corresponding to the upper dashed line. Similarly, using a one-particle amplitude for the upper blob in Fig. 1 one recovers the unitarity cuts corresponding to the lower dashed line in Fig. 3.

As a final point we emphasize that the unconventional terms found in earlier work[3,4] to be

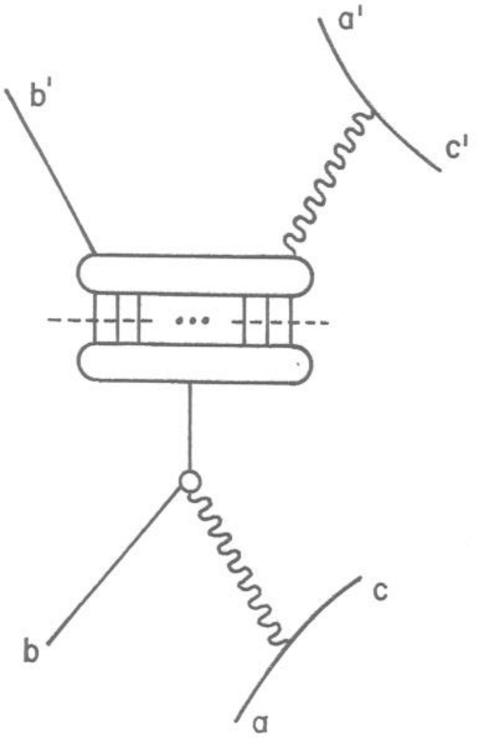

FIG. 6. One of the contributions to Fig. 1.

required for reasons of analyticity and to ensure Regge-cut cancellations are shown here to be needed for an apparently different reason. Here they are needed for detailed self-consistency of asymptotic power behavior in unitarity. We thus have two reasons for the presence of such terms arising out of self-consistent unitarity, analyticity, and Regge-pole behavior.

The internal consistency of such terms adds support for hope that the planar pure-pole bootstrap without Regge cuts, may, indeed, exist. But the present discussion carries the lesson that we must allow the properties of the Reggeon amplitudes to be freely determined and reinforced by the self-consistency conditions. A detailed model incorporating analyticity requirements only (such as the dual resonance model) is not *a priori* self-consistent. In the spirit of a true bootstrap theory, the conditions of self-consistency through unitarity in this example seem capable of generating and determining the properties of amplitudes without outside help.

ACKNOWLEDGMENT

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[1]G. Veneziano, Nucl. Phys. B74, 365 (1974); Phys. Lett. 52B, 220 (1974).
[2]For a review see, G. Chew and C. Rosenzweig, Phys. Rep. (to be published).
[3]J. R. Freeman and C. E. Jones, Phys. Rev. D 16, 1932 (1977).
[4]J. R. Freeman and C. E. Jones, Phys. Rev. D 16, 1936 (1977).
[5]J. R. Freeman, G. Veneziano, and Y. Zarmi, Nucl. Phys. B120, 477 (1977).
[6]C. Rosenzweig and G. Veneziano, Phys. Lett. 52B, 335 (1974).
[7]S. Mandelstam, Nuovo Cimento 30, 1113 (1963); 30, 1127 (1963); 30, 1148 (1963).