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Examining tensors in the lab: The dielectric permittivity and electrical resistivity of wood

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I. INTRODUCTION

Exposure to tensors in physics on an undergraduate level is primarily in the context of mechanics—a theoretical (i.e., no labs) treatment of the inertia tensor. Tensor properties of anisotropic materials are usually considered an advanced topic—both theoretically and experimentally. In presenting material properties, the focus is on isotropic properties, and only in passing is mention made that, for anisotropic materials, the electrical and thermal properties—permittivities, susceptibilities, conductivities, resistivities—are second rank symmetric tensors under a rotation of the coordinate axes.

Our aim here is to show that tensor properties can in fact be examined on an introductory level in the lab. The usual difficulties in working with anisotropic crystals—accessibility, cutting, polishing, cost—are overcome by using dry, straight-layer wood[1] (pine or cedar), a low-cost, accessible, easy-to-use anisotropic dielectric material. A further advantage of wood is its uniaxial symmetry. This considerably simplifies the technical analysis yet manifests the essential tensorial features of anisotropic crystal physics.

Dielectric permittivities and electrical resistivities of wood can be readily measured and analyzed in tensor terms. Our experiments make possible a ''hands-on'' approach that facilitates understanding tensor behavior. It should be emphasized that our primary purpose is not a detailed examination and analysis of the dielectric properties of wood but rather a broad outline of a new approach toward learning about tensor properties. Hence we expect that the experimental techniques and analysis given here can be refined and expanded.

II. QUALITATIVE DEMONSTRATION

A calcite crystal suspended in a homogeneous electric field turns to align its optic axis with the field direction. The trouble with this demonstration is that it is not convincing that the anisotropic crystal structure is behind the effect. An isotropic glass rod suspended in a homogeneous electric field also rotates to align the rod axis with the field direction.

What is needed is a spherically shaped anisotropic material with the appropriate dielectric properties. Dry, straight-layer wood suits this purpose well. Wood behaves die[...] cally like a uniaxial crystal with infinite symmetry ab[...] optic axis (the wood fiber axis). The crystal symmet[...] is: $\infty/m \cdot m$.

Suspend a wood ball (diameter 7.5 cm; fiber axis p[...] dicular to the line of suspension) between two parallel [...] plates charged to a high relative potential by a Wi[...] machine or other electrostatic generator. The ball rot[...] align the fiber axis with the electric field direction. C[...] the polarization properties of the anisotropic dielectr[...] crucial. One can expect that, for wood, the dielectric pe[...] tivity or equivalently, the dielectric susceptibility is [...] scalar constant but depends on the direction in the wo[...]

III. DIELECTRIC PERMITTIVITY

We now examine how the dielectric permittivity in [...] depends on the relative orientation of the wood fiber [...] and the direction of an applied electric field.

Make a condenser from two semi-cylinder metal pla[...] cutting in two (along a diameter) a metal cylinder (heig[...] cm, diameter 8 cm). The cylindrical cavity can be em[...] filled with an isotropic dielectric such as a paraffin cy[...] or with an anisotropic dielectric such as a wood cy[...] whose axis is perpendicular to the fiber direction.

The dielectric permittivity of air is well-approxima[...] the vacuum permittivity

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}.$$

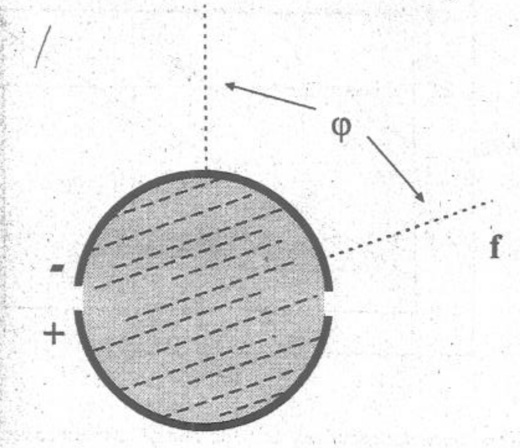
The relative permittivity (or dielectric coefficient) of pa[...] is known to be

$$\epsilon_p / \epsilon_0 = 2.26.$$

The permittivity of wood can be found from a series [...] pacitance measurements. The overall measured capac[...] C with a general dielectric of permittivity ϵ in the conde[...] is given by

$$C = (\epsilon / \epsilon_0) C' + C_0,$$

where C' is the unknown condenser capacitance and [...] the unknown background capacitance.



l. A wood cylinder between semi-cylinder condenser plates. The angle ecifies the wood fiber orientation.

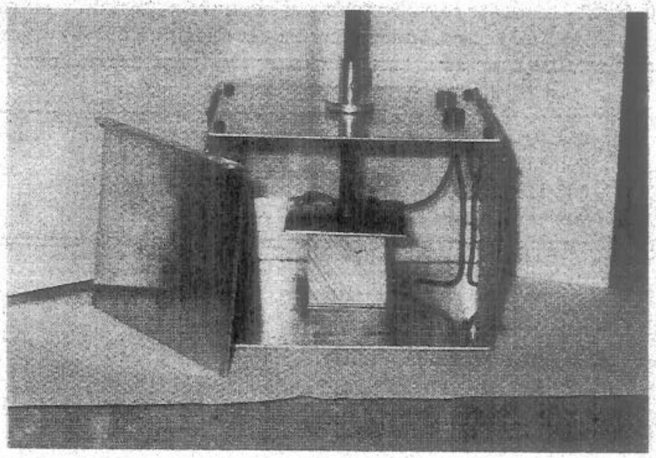


Fig. 2. A wood cube between parallel condenser plates surrounded by metal shielding.

ith just air in the condenser,

$$C(\text{air}) \approx C' + C_0. \tag{4}$$

h the paraffin cylinder, the capacitance is

$$C(\text{paraffin}) = (\epsilon_p/\epsilon_0)C' + C_0. \tag{5}$$

ince the value of (ϵ_p/ϵ_0) is known, measuring $C(\text{air})$ $C(\text{paraffin})$ allows the determination of C_0 and C'. w for a wood cylinder whose fibers are oriented at an le φ with respect to the plane bisecting the semi-nders—as shown in Fig. 1, the capacitance is

$$C(\text{wood}) = (\epsilon(\varphi)/\epsilon_0)C' + C_0. \tag{6}$$

hat measuring the capacitance $C(\text{wood})$ for a given le φ yields the relative permittivity $(\epsilon(\varphi)/\epsilon_0)$.

his strategy is simple but implementing it requires mea-ng capacitances to a resolution of tenths of picofarads. need for this scale of measurement can be seen from the acitance of a dielectric cube of permittivity ϵ:

$$\epsilon \times \text{cube face area/cube edge length}. \tag{7}$$

a cube with 3-cm edges, this capacitance becomes 655) (ϵ/ϵ_0) pF.

here are costly, off-the-shelf, capacitance meters with pi-rad resolution. But a straightforward and cost-effective native is to build your own device based on keeping tant the frequency in an LC resonant circuit which in-es the unknown capacitance and a calibrated variable acitor. The construction of a low-cost (under $100), pF resolution, capacitance meter is detailed in Ref. 2. We pted this design to build a meter with an operating fre-ncy of 10 kHz and with a scale from 0.1 to 10 pF.

he dielectric coefficient (ϵ/ϵ_0) of our wood cylinder for ral values of the angle φ are listed in Table I. The wood permittivity is clearly direction dependent. But the electric field between the semi-cylinder plates is not homogeneous. This complicates analyzing these results in tensor terms.

I. The relative permittivity of a wood cylinder for several values of ngle φ of Fig. 1.

φ	$\epsilon(\varphi)/\epsilon_0$
0°, 180°	2.75
30°, 150°	2.67
60°, 120°	2.40
90°	2.18

To overcome this problem, the preceding experiment is easily modified so that the condenser is a set of parallel metal plates (6 cm by 6 cm). Between the plates are put a series of wood cubes (4 cm on edge) whose fiber axis makes an angle θ=0°, 30°, 60°, 90° with respect to one of the cube faces. The cubes should be cut close together from the same block of wood since the dielectric properties are sample dependent due to the natural inhomogeneity of wood. The entire system is shielded by metal, as shown in Fig. 2.

Using a capacitance meter with a tenth of a picofarad resolution, we determined the wood permittivity $\epsilon(\theta)$ for various values of the angle θ between the applied homogeneous electric field **E** (perpendicular to the condenser plates) and the wood fiber axis **f**. See Fig. 3. The data are shown in Table II.

Now we relate $\epsilon(\theta)$ to the dielectric permittivity tensor ϵ_{ij}, a second-rank symmetric tensor[3] associated with the pair of linearly related vectors—the applied electric field intensity **E**, and the electric displacement **D**. Specify the coordinate axes by orthogonal unit vectors $\underline{1}$, $\underline{2}$, $\underline{3}$. Vectors can be decomposed into components along these coordinate axes:

$$\mathbf{E} = E_1\underline{1} + E_2\underline{2} + E_3\underline{3} \equiv (E_1, E_2, E_3), \tag{8}$$

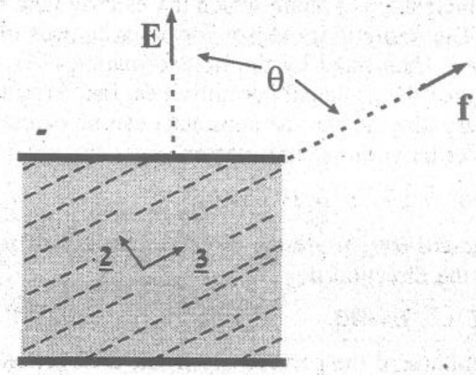


Fig. 3. The wood fiber direction **f** (parallel to the $\underline{3}$ axis) makes an angle θ with respect to the homogeneous electric field intensity **E** between parallel condenser plates.

Table II. The relative permittivity of wood cubes whose fiber axis makes an angle θ with the applied, homogeneous electric field.

θ	$\epsilon(\theta)/\epsilon_0$
0°, 180°	2.96
30°, 150°	2.75
60°, 120°	2.52
90°	2.25

$$\mathbf{D}=D_1\underline{\mathbf{1}}+D_2\underline{\mathbf{2}}+D_3\underline{\mathbf{3}}\equiv(D_1,D_2,D_3).$$

For isotropic materials the permittivity ϵ is a scalar quantity fixed by

$$\mathbf{D}=\epsilon\mathbf{E}; \tag{9}$$

but for anisotropic materials **D** and **E** are not necessarily parallel and are related via the permittivity tensor ϵ_{ij}:

$$D_i=\Sigma_j\epsilon_{ij}E_j,\quad i,j=1,2,3, \tag{10}$$

where the summation runs from 1 to 3.

Under an orthogonal transformation of the coordinate axes (say, $\underline{\mathbf{1}}$, $\underline{\mathbf{2}}$, $\underline{\mathbf{3}}$, rotates to $\underline{\mathbf{1}'}$, $\underline{\mathbf{2}'}$, $\underline{\mathbf{3}'}$,) the "new" position coordinates (i.e., in the rotated system) r_1', r_2', r_3' are related to the "old" position coordinates (i.e., in the unrotated system) r_1, r_2, r_3 by

$$r_1'=\Sigma_j a_{ij}r_j,\quad a_{ij}\equiv\underline{\mathbf{i}'}\cdot\underline{\mathbf{j}},\quad i,j=1,2,3, \tag{11}$$

where the a_{ij} are the direction cosines of the new axes relative to the old axes. This transformation applies to the first-rank tensor quantities **E** and **D**.

The permittivity tensor in the new coordinate system ϵ_{ij}' is related to the permittivity tensor in the old coordinate system ϵ_{ij} by

$$\epsilon_{ij}'=\Sigma_k\Sigma_l a_{ik}a_{jl}\epsilon_{kl},\quad i,j,k,l=1,2,3. \tag{12}$$

This transformation defines the permittivity as a second-rank tensor.

There can be found a particular set of coordinate axes—the principal axes—where a symmetric second-rank tensor has only diagonal components. In the principal system, the permittivity tensor becomes

$$\epsilon_{ij}=\epsilon_i\delta_{ij},\quad i,j=1,2,3, \tag{13}$$

where a principal permittivity ϵ_i is measured in the direction of the principal axis $\underline{\mathbf{i}}$ along which the electric field has been applied. The permittivity tensor for other choices of coordinate axes is then fixed by the transformation (12). In other words, given the principal permittivities, the permittivity for an arbitrary direction in the substance can be calculated.

Consider an arbitrary unit vector

$$\mathbf{n}=\cos\theta_1\underline{\mathbf{1}}+\cos\theta_2\underline{\mathbf{2}}+\cos\theta_3\underline{\mathbf{3}}, \tag{14}$$

where the $\cos\theta_i\equiv\underline{\mathbf{i}}\cdot\mathbf{n}$ are the direction cosines of **n**. Applying **E** in the direction **n**,

$$\mathbf{E}=E\mathbf{n},\quad E\equiv|\mathbf{E}|. \tag{15}$$

The magnitude of the permittivity in the **n** direction, $\epsilon[\mathbf{n}]$, is just the component of **D**/E along **n**:

$$\epsilon[\mathbf{n}]=\mathbf{D}\cdot\mathbf{n}/E=\Sigma_i\Sigma_j\epsilon_{ij}E_j\cos\theta_i/E$$
$$=\Sigma_i\Sigma_j\epsilon_{ij}\cos\theta_i\cos\theta_j. \tag{16}$$

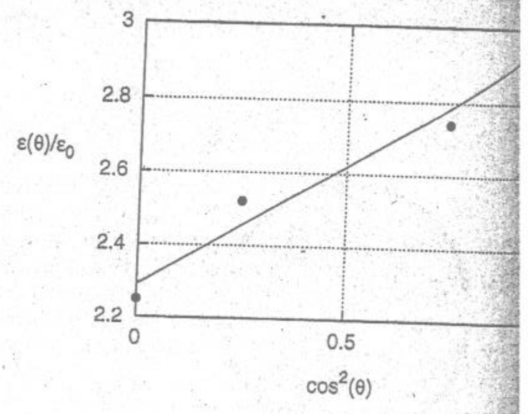


Fig. 4. Wood relative permittivity as a function of $\cos^2\theta$ togethe "least-squares" linear fit. Experimental values are shown as dots.

This result also follows directly from the tensor trans tion law (12). Suppose the new coordinate axes are ch that $\underline{3'}$ is along **n**. Then for the electric field along permittivity along $\underline{3'}$ is

$$\epsilon[\mathbf{n}]=\epsilon_{33}'=\Sigma_i\Sigma_j a_{3i}a_{3j}\epsilon_{ij}$$
$$=\Sigma_i\Sigma_j(\underline{3'}\cdot\underline{\mathbf{i}})(\underline{3'}\cdot\underline{\mathbf{j}})\epsilon_{ij}$$
$$=\Sigma_i\Sigma_j\cos\theta_i\cos\theta_j\epsilon_{ij},$$

as in Eq. (16).

For the principal system, the permittivity in the **n** di simplifies to

$$\epsilon[\mathbf{n}]=\Sigma_i(\cos\theta_i)^2\epsilon_i.$$

A further simplification follows for a uniaxial syste symmetry axis is a principal axis (say it is the $\underline{3}$ axis) a two remaining principal axes $\underline{1}$, $\underline{2}$ can be chosen as an orthogonal unit vectors in the plane perpendicular Since, for wood, there is ∞-fold symmetry about $\underline{3}$, it vious that the permittivity perpendicular to the sym axis is the same in any direction. Thus

$$\epsilon_3=\epsilon(\theta=0°),\quad\epsilon_1=\epsilon_2=\epsilon(\theta=90°).$$

When the electric field **E** is directed along a principal a the wood, the displacement field **D** is parallel to **E** b permittivity along the fiber direction is different tha perpendicular to the fibers.

For the configuration of Fig. 3, the electric field is

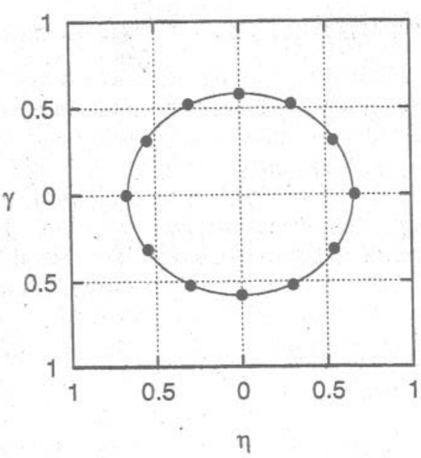
$$\mathbf{E}=(0,E\cos(90°-\theta),E\cos\theta),$$

so that by Eqs. (18) and (19)

$$\epsilon(\theta)=\epsilon_2\cos^2(90°-\theta)+\epsilon_3\cos^2\theta,$$
$$\epsilon(\theta)=\epsilon(90°)\sin^2\theta+\epsilon(0°)\cos^2\theta.$$

This simple result can also be "guessed directly." The mittivity transforms as a second-rank symmetric tens $\epsilon(\theta)$ is expected to be a second-order polynomial in the formation direction cosines. Because of the uniaxial sy try, the polynomial can be built only from $\cos\theta$ or co $-\theta)$. The physical requirement that $\epsilon(\theta)$ must be eve gives the result of Eq. (21).

Figure 4 shows the measured wood dielectric coeffi $\epsilon(\theta)/\epsilon_0$, as a function of $\cos^2\theta$ together with a least-sq linear fit consistent with Eq. (21).



5. The wood permittivity ellipse of Eq. (23) where the semi-major axis emi-minor axis are fixed by the linear fit of Fig. 4. The radial length at gle θ is $1/[\epsilon(\theta)/\epsilon_0]^{1/2}$. Experimental values are shown as dots.

geometrical representation[4] for second-rank tensor erties can be applied to the wood permittivity by introng into Eq. (21) the variables

$$\gamma \equiv \cos\,\theta/[\epsilon(\theta)/\epsilon_0]^{1/2}, \quad \eta \equiv \sin\,\theta/[\epsilon(\theta)/\epsilon_0]^{1/2}. \qquad (22)$$

result is an ellipse equation—the permittivity ellipse—

$$\eta^2/a^2+\gamma^2/b^2=1 \qquad (23)$$

semi-major axis, $a\equiv 1/[\epsilon(90°)/\epsilon_0]^{1/2}$, and semi-minor $b\equiv 1/[\epsilon(0°)/\epsilon_0]^{1/2}$. The radius distance $d(\theta)$ from the se center to a point on the ellipse making an angle θ with ect to the γ axis is just

$$d(\theta)=(\gamma^2+\eta^2)^{1/2}=1/[\epsilon(\theta)/\epsilon_0]^{1/2}. \qquad (24)$$

he relative permittivity at an angle θ with respect to the axis is related to the inverse square of the radius of the ittivity ellipse at an angle θ with respect to the γ axis. re 5 converts the $\epsilon(\theta)/\epsilon_0$ curve of Fig. 4 to the permity ellipse of Eq. (23).

he Appendix discusses an entirely different way of findthe principal dielectric permittivities of wood based on d's natural birefringence at microwave frequencies.

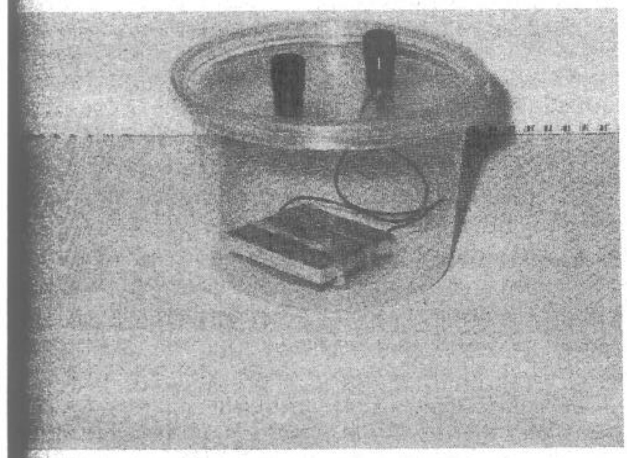


Fig. 6. Thin wood block between parallel metal plates.

Table III. The electrical resistivity of wood cubes whose fiber axis makes an angle θ with the applied, homogeneous electric field.

θ	$\rho(\theta)$ (10^9 Ω m)
0°, 180°	1.0
30°, 150°	1.3
60°, 120°	3.2
90°	3.5

IV. ELECTRICAL CONDUCTIVITY/RESISTIVITY

For isotropic materials obeying Ohm's law,

$$\mathbf{j}=\sigma\mathbf{E}, \qquad (25)$$

where $\mathbf{E}\equiv(E_1,E_2,E_3)$ is the applied electric field vector, $\mathbf{j}\equiv(j_1,j_2,j_3)$ is the resulting current density vector (the current I per unit cross section A perpendicular to the current), and σ is the electrical conductivity. This may be inverted to

$$\mathbf{E}=\rho\mathbf{j}, \quad \rho=1/\sigma, \qquad (26)$$

where ρ is the electrical resistivity.

For anisotropic crystals, the electrical conductivity is a second-rank symmetric tensor σ_{ik} associated with vectors **E** and **j**:

$$j_i=\Sigma_k\sigma_{ik}E_k, \quad i=1,2,3. \qquad (27)$$

Alternatively,

$$E_i=\Sigma_k\rho_{ik}j_k, \qquad (28)$$

where ρ_{ik} is the electrical resistivity tensor.

The experimental examination of the conductivity tensor is not common, as W. A. Wooster[5] has pointed out: "Electric conductivity is likewise a second-order tensor but consideration of it is omitted from this book because of the difficulty of obtaining suitable crystals in which the electric conductivity can be measured." But there is no such difficulty in the case of wood.

Our samples are thin wood blocks with smooth faces (4.5 cm by 4.5 cm and thickness $t=6$ mm). The fiber axis orientation makes an angle $\theta=0°$, 30°, 60°, 90° with respect to the direction perpendicular to the square faces. Each block is sandwiched between two thin metal plates (4.5 cm by 4.5 cm) as shown in Fig. 6. The wood–metal contact can be made uniform by coating the wood with graphite. A stable dc

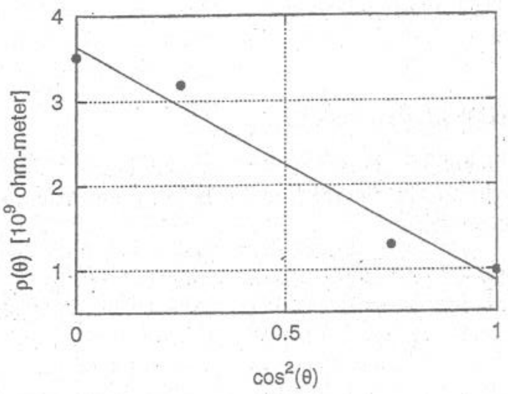


Fig. 7. Wood resistivity as a function of $\cos^2\theta$ together with a "least squares" linear fit.

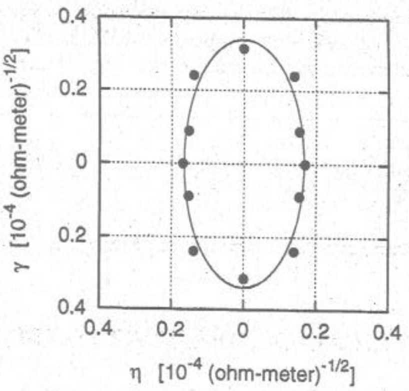


Fig. 8. The wood resistivity ellipse of Eq. (33). The semi-major and semi-minor axes are fixed by the linear fit of Fig. 7. The radial length at an angle θ with respect to the γ axis is $[1/\rho(\theta)]^{1/2}$. Experimental values are shown as dots.

voltage V is applied and the resulting current $I(\theta)$ through the wood block is measured. For $V=20$ V, the current was on the scale of 10^{-8} A but higher voltages raise the current.

The resistance $R(\theta)$ of a block whose fiber axis makes an angle θ with the homogeneous electric field **E** between the plates is

$$V/I(\theta)=R(\theta)=\rho(\theta)t/A. \tag{29}$$

So the resistivity $\rho(\theta)$ can be found as a function of θ. Data for $V=400$ V, is shown in Table III.

The relation of $\rho(\theta)$ to the resistivity tensor, ρ_{ik}, is analogous to that of the permittivity $\epsilon(\theta)$ to the permittivity tensor. In the principal system, resistivities for our uniaxial system are

$$\rho_{ik}=\delta_{ik}\rho_i, \quad i,k=1,2,3, \tag{30}$$

so that if the fiber direction is along $\underline{3}$, the principal wood resistivities are

$$\rho_3=\rho(0°), \quad \rho_1=\rho_2=\rho(90°). \tag{31}$$

As in Eq. (21), the angular dependence of the resistivity is fixed by second-rank tensor behavior under uniaxial symmetry:

$$\rho(\theta)=\rho(90°)\sin^2\theta+\rho(0°)\cos^2\theta. \tag{32}$$

Figure 7 shows the wood resistivity data as a function of $\cos^2\theta$ together with a "least-squares" linear fit. The corresponding resistivity ellipse,

$$\rho(90°)\eta^2+\rho(0°)\gamma^2=1, \tag{33}$$

is shown in Fig. 8.

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APPENDIX

There is another method that can be used to determine principal permittivities of wood, but its theoretical basis on an advanced application of electromagnetic theory. It be shown[6] that Maxwell's equations for anisotropic me (i.e., where the permittivity is a second-rank tensor) give direction-dependent electromagnetic wave speed in the dium. Hence, the refractive index in the medium will also anisotropic. For any direction in a naturally birefringent dium, the refractive index can be summarized by indicatrix—the ellipsoid:

$$x^2/n_1^2+y^2/n_2^2+z^2/n_3^2=1,$$

where x, y, z are directed along the principal axes for electric permittivity and the principal refractive indices n_2, n_3 are related to the principal permittivities ϵ_1, ϵ by

$$n_i=(\epsilon_i/\epsilon_0)^{1/2}.$$

Although the refractive index is not a tensor, its direc dependence is set by the permittivity which is a tensor.

Measurement of the principal refractive indices yields, by (A2), the principal permittivities.

For the case of uniaxial symmetry in the z direction indicatrix is an ellipsoid of revolution about the symm axis and

$$n_1=n_2=n_o\equiv\text{ordinary refractive index},$$

$$n_3=n_e\equiv\text{extraordinary refractive index}.$$

In Ref. 1, we outlined a Young's apparatus method microwaves for measuring the refractive index of woo various directions relative to the fiber axis. This techn can be used to find n_e and n_o for wood and hence the indicatrix. The extraordinary index of refraction for vari angles with respect to the symmetry axis can then be ca lated from the indicatrix and compared with the meas value.

The principal dielectric permittivities for wood can found from n_o and n_c. But since permittivity is freque dependent, the permittivities at 10-GHz microwave frequ cies are not identical to those found by the capacitance m surements at 10 kHz.

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[1]The anisotropy of wood was used to examine natural birefringence S. Perkalskis and J. R. Freeman, "Demonstrating crystal optics usi crowaves on wood targets," Am. J. Phys. **63** (8), 762–764 (1995).

[2]E. Vladkov, "*C*-meter Resolves to 0.1 pF," Electronics World August 1996), pp. 552–556.

[3]An excellent introduction to tensors for crystal physics is: J. F. Nye cal Properties of Crystals (Oxford U.P., Oxford, 1976), Chaps. I

[4]Reference 3, p. 26.

[5]W. A. Wooster, *Experimental Crystal Physics* (Clarendon, Oxford p. 48.

[6]Reference 3, Appendix H.