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The Geometry of Lightlike Hypersurfaces of the de Sitter Space

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Abstract. It is proved that the geometry of lightlike hypersurfaces of the de Sitter space S_1^{n+1} is directly connected with the geometry of hypersurfaces of the conformal space C^n. This connection is applied for a construction of an invariant normalization and an invariant affine connection of lightlike hypersurfaces as well as for studying singularities of lightlike hypersurfaces.

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0. Introduction

The projective model of the non-Euclidean geometry (the Cayley–Klein model) is closely connected with models of conformal geometry and the geometry of the de Sitter space. In fact, the hyperbolic space H^{n+1} of dimension $n+1$ – the Lobachevsky space – admits a mapping onto internal domain of an n-dimensional oval hyperquadric Q^n of a projective space P^{n+1}. On this hyperquadric itself the geometry of an n-dimensional conformal space C^n is realized, and outside of the hyperquadric Q^n the geometry of the $(n+1)$-dimensional de Sitter space S_1^{n+1} is realized. Moreover, the group of projective transformations of the space P^{n+1} keeping the hyperquadric Q^n invariant and transferring its internal domain into itself (this group is denoted by $\mathbf{PO}(n+2, 1)$ – see [7, p. 7]) is isomorphic to the group of motions of the Lobachevsky space H^{n+1}, the conformal space C^n, and the de Sitter space S_1^{n+1}. It is clear that there exist deep connections among these three geometries.

The Lobachevsky geometry is the first example of geometry which differs from the Euclidean geometry. Numerous books and papers are devoted to the Lobachevsky geometry. Conformal differential geometry was also studied in detail. In particular, it was studied in the last authors' book [7]. In spite of the fact that the geometry of the de Sitter space is the simplest model of spacetime of general relativity, this geometry was not studied thoroughly. The de Sitter space sustains the Lorentzian metric of constant positive curvature.

In the present paper we study the geometry of the de Sitter space S_1^{n+1} using its connection with the geometry of the conformal space. We prove that the geometry of lightlike hypersurfaces of the space S_1^{n+1}, which play an important role in general relativity (see the book [10]), is directly connected with the geometry of hypersurfaces of the conformal space C^n. The latter was studied in detail in the papers of the first author (see [1–5]) and also in the book [7]. This simplifies the study of lightlike hypersurfaces of the de Sitter space S_1^{n+1} and makes possible to apply for their consideration the apparatus constructed in the conformal theory.

In Section 1 we study the geometry of the de Sitter space and its connection with the geometry of the conformal space. After this we study lightlike hypersurfaces U^n in the space S_1^{n+1}, investigate their structure, and prove that such a hypersurface is tangentially degenerate of rank $r \leqslant n-1$. Its rectilinear or plane generators form an isotropic fiber bundle on U^n.

In Sections 2–5 we investigate lightlike hypersurfaces U^n of maximal rank, and for their study we use the relationship between the geometry of such hypersurfaces and the geometry of hypersurfaces of the conformal space. For a lightlike hypersurface, we construct the fundamental quadratic forms and connections determined by a normalization of a hypersurface by means of a distribution (the screen distribution) which is complementary to the isotropic distribution. The screen distribution plays an important role in the book [10] since it defines a connection on a lightlike hypersurface U^n, and it appears to be important for applications. We prove that the screen distribution on a lightlike hypersurface can be constructed invariantly by means of quantities from a third-order differential neighborhood, that is, such a distribution is intrinsically connected with the geometry of a hypersurface.

In Section 5 we study singular points of a lightlike hypersurface in the de Sitter space S_1^{n+1}, classify them, and describe the structure of hypersurfaces carrying singular points of different types. Moreover, we establish the connection of this classification with that of canal hypersurfaces of the conformal space.

In Section 6 we consider lightlike hypersurfaces of reduced rank. Such hypersurfaces carry lightlike rectilinear generators along which their tangent hyperplanes are constant. For such hypersurfaces, again in a third-order differential neighborhood, we construct an invariant screen distribution and an invariant affine connection. However, the method of construction is different from that for lightlike hypersurfaces of maximal rank, since the construction used for hypersurfaces of maximal rank fails for hypersurfaces of reduced rank. We establish a connection of lightlike hypersurfaces of reduced rank with quadratic hyperbands of a multidimensional projective space.

The principal method of our investigation is the method of moving frames and exterior differential forms in the form in which it is presented in the books [6] and [7]. All functions considered in the paper are assumed to be real and differentiable, and all manifolds are assumed to be smooth with the possible exception of some isolated singular points and singular submanifolds.

1. The de Sitter Space

1. In a projective space P^{n+1} of dimension $n+1$ we consider an oval hyperquadric Q^n. Let x be a point of the space P^{n+1} with projective coordinates $(x^0, x^1, \dots, x^{n+1})$. The hyperquadric Q^n is determined by equations

$$(x, x) := g_{\xi\eta} x^\xi x^\eta = 0, \quad \xi, \eta = 0, \dots, n+1, \tag{1}$$

whose left-hand side is a quadratic form (x, x) of signature $(n+1, 1)$. The hyperquadric Q^n divides the space P^{n+1} into two parts, external and internal. Normalize the quadratic form (x, x) in such a way that for the points of the external part the inequality $(x, x) > 0$ holds. This external domain is a model of the *de Sitter space* S_1^{n+1} (see [15]). We will identify the external domain of Q^n with the space S_1^{n+1}. The hyperquadric Q^n is the *absolute* of the space S_1^{n+1}.

On the hyperquadric Q^n of the space P^{n+1} the geometry of a conformal space C^n is realized. The bijective mapping $C^n \leftrightarrow Q^n$ is called the *Darboux mapping*, and the hyperquadric Q^n itself is called the *Darboux hyperquadric*.

Under the Darboux mapping to hyperspheres of the space C^n, there correspond cross-sections of the hyperquadric Q^n by hyperplanes ξ. But to a hyperplane ξ there corresponds a point x that is polar-conjugate to ξ with respect to Q^n and lies outside of Q^n, that is, a point of the space S_1^{n+1}. Thus, to hyperspheres of the space C^n there correspond points of the space S_1^{n+1}.

Let x be an arbitrary point of the space S_1^{n+1}. The tangent lines from the point x to the hyperquadric Q^n form a second-order cone C_x with vertex at the point x. This cone is called the *isotropic cone*. For spacetime whose model is the space S_1^{n+1} this cone is the light cone, and its generators are lines of propagation of light impulses whose source coincides with the point x.

The cone C_x separates all straight lines passing through the point x into spacelike (not having common points with the hyperquadric Q^n), timelike (intersecting Q^n in two different points), and lightlike (tangent to Q^n). The lightlike straight lines are generators of the cone C_x.

To a spacelike straight line $l \subset S_1^{n+1}$ there corresponds an elliptic pencil of hyperspheres in the conformal space C^n. All hyperspheres of this pencil pass through a common $(n-2)$-sphere S^{n-2} (the center of this pencil). The sphere S^{n-2} is the intersection of the hyperquadric Q^n and the $(n-1)$-dimensional subspace of the space P^{n+1} which is polar-conjugate to the line l with respect to the hyperquadric Q^n.

To a timelike straight line $l \subset S_1^{n+1}$ there corresponds a hyperbolic pencil of hyperspheres in the space C^n. Two arbitrary hyperspheres of this pencil do not have common points, and the pencil contains two hyperspheres of zero radius which correspond to the points of intersection of the straight line l and the hyperquadric Q^n.

Finally, to a lightlike straight line $l \subset S_1^{n+1}$ there corresponds a parabolic pencil of hyperspheres in the space C^n consisting of hyperspheres tangent one to another at a point, that is, a unique hypersphere of zero radius belonging to this pencil.

Hyperplanes of the space S_1^{n+1} are also divided into three types. Spacelike hyperplanes do not have common points with the hyperquadric Q^n; a timelike hyperplane intersects Q^n along a real hypersphere; and lightlike hyperplanes are tangent to Q^n. Subspaces of any dimension r, $2 \leqslant r \leqslant n-1$, can be also classified in a similar manner.

Let us apply the method of moving frames to study some questions of differential geometry of the space S_1^{n+1}. With a point $x \in S_1^{n+1}$ we associate a family of projective frames $\{A_0, A_1, \ldots, A_{n+1}\}$. However, in order to apply formulas derived in the book [7], we will use the notations used in it. Namely, we denote by A_n the vertex of the moving frame which coincides with the point x, $A_n = x$; we locate the vertices A_0, A_i $(i = 1, \ldots, n-1)$, and A_{n+1} at the hyperplane ξ which is polar-conjugate to the point x with respect to the hyperquadric Q^n, and we assume that the points A_0 and A_{n+1} lie on the hypersphere $S^{n-1} = Q^n \cap \xi$, and the points A_i are polar-conjugate to the straight line $A_0 A_{n+1}$ with respect to S^{n-1}. Since $(x, x) > 0$, we can normalize the point A_n by the condition $(A_n, A_n) = 1$. The points A_0 and A_{n+1} are not polar-conjugate with respect to the hyperquadric Q^n. Hence, we can normalize them by the condition $(A_0, A_{n+1}) = -1$. As a result, the matrix of scalar products of the frame elements has the form

$$(A_\xi, A_\eta) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & g_{ij} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i, j = 1, \ldots, n-1, \tag{2}$$

and the quadratic form (x, x) takes the form

$$(x, x) = g_{ij} x^i x^j + (x^n)^2 - 2x^0 x^{n+1}. \tag{3}$$

The quadratic form $g_{ij} x^i x^j$ occurring in (3) is positive definite.

The equations of infinitesimal displacement of the conformal frame $\{A_\xi\}$, $\xi = 0, 1, \ldots, n+1$, we have constructed have the form

$$\mathrm{d}A_\xi = \omega_\xi^\eta A_\eta, \quad \xi, \eta = 0, 1, \ldots, n+1, \tag{4}$$

where by (2), the 1-forms ω_ξ^η satisfy the following Pfaffian equations:

$$\left\{\begin{array}{ll} \omega_0^{n+1} = \omega_{n+1}^0 = 0, & \omega_0^0 + \omega_{n+1}^{n+1} = 0, \\ \omega_i^{n+1} = g_{ij}\omega_0^j, & \omega_i^0 = g_{ij}\omega_{n+1}^j, \\ \omega_n^{n+1} - \omega_0^n = 0, & \omega_n^0 - \omega_{n+1}^n = 0, \\ g_{ij}\omega_n^j + \omega_i^n = 0, & \omega_n^n = 0, \\ dg_{ij} = g_{jk}\omega_i^k + g_{ik}\omega_j^k. & \end{array}\right. \tag{5}$$

These formulas are precisely the formulas derived in the book [7] (see p. 32) for the conformal space C^n.

It follows from (4) that

$$dA_n = \omega_n^0 A_0 + \omega_n^i A_i + \omega_n^{n+1} A_{n+1}. \tag{6}$$

The differential dA_n belongs to the tangent space $T_x(S_1^{n+1})$, and the 1-forms ω_n^0, ω_n^i, and ω_n^{n+1} form a coframe of this space. The total number of these forms is $n+1$, and this number coincides with the dimension of $T_x(S_1^{n+1})$. The scalar square of the differential dA_n is the metric quadratic form $\widetilde{g}$ on the manifold S_1^{n+1}. By (2), this quadratic form $\widetilde{g}$ can be written as

$$\widetilde{g} = (dA_n, dA_n) = g_{ij}\omega_n^i \omega_n^j - 2\omega_n^0 \omega_n^{n+1}.$$

Since the first term of this expression is a positive definite quadratic form, the form $\widetilde{g}$ is of Lorentzian signature $(n, 1)$. The coefficients of the form $\widetilde{g}$ produce the metric tensor of the space S_1^{n+1} whose matrix is obtained from the matrix (2) by deleting the nth row and the nth column.

The quadratic form $\widetilde{g}$ defines on S_1^{n+1} a pseudo-Riemannian metric of signature $(n, 1)$. The isotropic cone defined in the space $T_x(S_1^{n+1})$ by the equation $\widetilde{g} = 0$ coincides with the cone C_x that we defined earlier in the space S_1^{n+1} geometrically.

The 1-forms ω_ξ^η occurring in equations (4) satisfy the structure equations of the space C^n:

$$d\omega_\xi^\eta = \omega_\xi^\zeta \wedge \omega_\zeta^\eta, \tag{7}$$

which are obtained by taking exterior derivatives of Equations (4) and which are conditions of complete integrability of (4). The forms ω_ξ^η are invariant forms of the fundamental group $\mathbf{PO}(n+2, 1)$ of transformations of the spaces H^{n+1}, C^n, and S_1^{n+1} which is locally isomorphic to the group $\mathbf{SO}(n+2, 1)$.

Let us write Equations (7) for the 1-forms ω_n^0, ω_n^i, and ω_n^{n+1} making up a coframe of the space $T_x(S_1^{n+1})$ in more detail:

$$\begin{aligned}
d\omega_n^0 &= \omega_n^0 \wedge \omega_0^0 + \omega_n^i \wedge \omega_i^0, \\
d\omega_n^i &= \omega_n^0 \wedge \omega_0^i + \omega_n^j \wedge \omega_j^i + \omega_n^{n+1} \wedge \omega_{n+1}^i, \\
d\omega_n^{n+1} &= \phantom{\omega_n^0 \wedge \omega_0^i + {}} \omega_n^i \wedge \omega_i^{n+1} + \omega_n^{n+1} \wedge \omega_{n+1}^{n+1}.
\end{aligned} \tag{8}$$

The last equations can be written in the matrix form as follows:

$$d\theta = -\omega \wedge \theta, \tag{9}$$

where $\theta = (\omega_n^u)$, $u = 0, i, n+1$, is the column matrix with its values in the vector space $T_x(S_1^{n+1})$, and $\omega = (\omega_v^u)$, $u, v = 0, i, n+1$, is a square matrix of order $n+1$ with values in the Lie algebra of the group of admissible transformations of coframes of the space $T_x(S_1^{n+1})$. The form ω is the connection form of the space S_1^{n+1}. In detail this form can be written as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_i^0 & 0 \\ \omega_0^i & \omega_i^j & \omega_{n+1}^i \\ 0 & \omega_i^{n+1} & \omega_{n+1}^{n+1} \end{pmatrix}. \tag{10}$$

By (5), in this matrix, only the forms in the left upper corner, which form an $n \times n$-matrix, are linearly independent.

The connection form (10) allows us to find the differential equation of geodesics in the space S_1^{n+1}. These lines coincide with straight lines of the ambient space P^{n+1}; more precisely, they coincide with the parts of these straight lines which lie outside of the Darboux hyperquadric Q^n. We will look for their equation in the form $x = x(t)$, and we will impose the vertex A_n of the moving frame with the point x, $A_n = x(t)$. Write the decomposition of the tangent vector to a geodesic in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi^u A_u, \quad u = 0, i, n+1.$$

For a geodesic, the second differential $\mathrm{d}^2x/\mathrm{d}t^2$ is collinear to its tangent vector $\mathrm{d}x/\mathrm{d}t$. This implies that

$$\frac{\mathrm{d}\xi^u}{\mathrm{d}t} A_u + \xi^v \omega_v^u A_u = \alpha \xi^u A_u,$$

where the connection 1-forms ω_v^u composing the matrix (10) are calculated along the curve $x = x(t)$, and α is a new 1-form. Hence, the differential equation of geodesics has the form

$$\frac{\mathrm{d}\xi^u}{\mathrm{d}t} + \xi^v \omega_v^u = \alpha \xi^u. \tag{11}$$

The same Equation (11) is the equation of straight lines of the space P^{n+1}.

Next we will find the curvature form and the curvature tensor of the space S_1^{n+1}. To this end, we take exterior derivative of the connection form ω, more precisely, of its independent part. Applying Equations (7), we find the following components of the curvature form:

$$\begin{cases} \Omega_0^0 = \mathrm{d}\omega_0^0 - \omega_0^i \wedge \omega_i^0 = \omega_n^{n+1} \wedge \omega_n^0, \\ \Omega_0^i = \mathrm{d}\omega_0^i - \omega_0^0 \wedge \omega_0^i - \omega_0^j \wedge \omega_j^i = \omega_n^{n+1} \wedge \omega_n^i, \\ \Omega_i^0 = \mathrm{d}\omega_i^0 - \omega_i^0 \wedge \omega_0^0 - \omega_i^j \wedge \omega_j^0 = -g_{ij}\omega_n^j \wedge \omega_n^0, \\ \Omega_j^i = \mathrm{d}\omega_j^i - \omega_j^0 \wedge \omega_0^i - \omega_j^k \wedge \omega_k^i - \omega_j^{n+1} \wedge \omega_{n+1}^i = -g_{jk}\omega_n^k \wedge \omega_n^i. \end{cases} \tag{12}$$

But the general expression of the curvature form of an $(n+1)$-dimensional pseudo-Riemannian space with a coframe ω_n^0, ω_n^i, and ω_n^{n+1} has the form

$$\Omega_s^r = \mathrm{d}\omega_s^r - \omega_s^t \wedge \omega_t^r = \frac{1}{2} R_{suv}^r \omega_n^u \wedge \omega_n^v, \tag{13}$$

where $r, s, t, u, v = 0, 1, \ldots, n-1, n+1$ (see, for example, [14]). Comparing Equations (12) and (13), we find that

$$\Omega_s^r = \omega_u^n \wedge g_{sv}\omega_n^v$$

and

$$R_{suv}^r = \delta_u^r g_{sv} - \delta_v^r g_{su}, \tag{14}$$

where (g_{sv}) is the matrix of coefficients of the quadratic form (2). But this means that the space S_1^{n+1} is a pseudo-Riemannian space of constant positive curvature $K = 1$. The Ricci tensor of this space has the form

$$R_{sv} = R_{srv}^r = n g_{sv}. \tag{15}$$

This confirms that the space S_1^{n+1}, as any pseudo-Riemannian space of constant curvature, is the Einstein space.

Thus by means of the method of moving frame we proved the following well-known theorem (see, for example, [14]):

THEOREM 1. *The de Sitter space, whose model is the domain of a projective space P^{n+1} lying outside of an oval hyperquadric Q^n, is a pseudo-Riemannian space of Lorentzian signature $(n, 1)$ and of constant positive curvature $K = 1$. This space is homogeneous, and its fundamental group* $\mathbf{PO}(n+2, 1)$ *is locally isomorphic to the special orthogonal group* $\mathbf{SO}(n+2, 1)$.

2. Lightlike Hypersurfaces in the de Sitter Space

A hypersurface U^n in the de Sitter space S_1^{n+1} is said to be *lightlike* if all its tangent hyperplanes are lightlike, that is, they are tangent to the hyperquadric Q^n which is the absolute of the space S_1^{n+1}.

Denote by x an arbitrary point of the hypersurface U^n, by η the tangent hyperplane to U^n at the point $x, \eta = T_x(U^n)$, and by y the point of tangency of the hyperplane η with the hyperquadric Q^n. Next, as in Section 1, denote by ξ the hyperplane which is polar-conjugate to the point x with respect to the hyperquadric Q^n, and associate with a point x a family of projective frames such that $x = A_n$, $y = A_0$, the points A_i, $i = 1, \ldots, n-1$, belong to the intersection of the hyperplanes ξ and η, $A_i \in \xi \cap \eta$, and the point A_{n+1}, as well as the point A_0, belong to the straight line that is polar-conjugate to the $(n-2)$-dimensional subspace spanned by the points A_i. In addition, we normalize the frame vertices in the same way as this was done in Section 1. Then the matrix of scalar products of the frame elements has the form (2), and the components of infinitesimal displacements of the moving frame satisfy the Pfaffian Equations (5).

Since the hyperplane η is tangent to the hypersurface U^n at the point $x = A_n$ and does not contain the point A_{n+1}, the differential of the point $x = A_n$ has the form

$$\mathrm{d}A_n = \omega_n^0 A_0 + \omega_n^i A_i, \tag{16}$$

the following equation holds:

$$\omega_n^{n+1} = 0, \tag{17}$$

and the forms ω_n^0 and ω_n^i are basis forms of the hypersurface U^n. By relations (5), it follows from Equation (16) that

$$\omega_0^n = 0 \tag{18}$$

and

$$\mathrm{d}A_0 = \omega_0^0 A_0 + \omega_0^i A_i. \tag{19}$$

Taking exterior derivative of Equation (17), we obtain

$$\omega_n^i \wedge \omega_i^{n+1} = 0.$$

Since the forms ω_n^i are linearly independent, by Cartan's lemma, we find from the last equation that

$$\omega_i^{n+1} = \nu_{ij}\omega_n^j, \quad \nu_{ij} = \nu_{ji}. \tag{20}$$

Applying an appropriate formula from (5), we find that

$$\omega_0^i = g^{ij}\omega_j^{n+1} = g^{ik}\nu_{kj}\omega_n^j, \tag{21}$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

Now formulas (16) and (19) imply that for $\omega_n^i = 0$, the point A_n of the hypersurface U^n moves along the isotropic straight line $A_n A_0$, and hence, U^n is a ruled hypersurface. In what follows, we assume that the *entire* straight line $A_n A_0$ belongs to the hypersurface U^n.

In addition, formulas (16) and (19) show that at any point of a generator of the hypersurface U^n, its tangent hyperplane is fixed and coincides with the hyperplane η. Thus, U^n is a *tangentially degenerate hypersurface*.

We recall that the *rank* of a tangentially degenerate hypersurface is the number of parameters on which the family of its tangent hyperplanes depends (see, for example, [6, p. 113]). From relations (16) and (19) it follows that the tangent hyperplane η of the hypersurface U^n along its generator $A_n A_0$ is determined by this generator and the points A_i, $\eta = A_n \wedge A_0 \wedge A_1 \wedge \cdots \wedge A_{n-1}$. The displacement of this hyperplane is determined by the differentials (16), (19), and

$$\mathrm{d}A_i = \omega_i^0 A_0 + \omega_i^j A_j + \omega_i^n A_n + \omega_i^{n+1} A_{n+1}.$$

But by (5), $\omega_i^n = -g_{ij}\omega_n^j$, and the forms ω_i^{n+1} are expressed according to formulas (20). From formulas (20) and (21) it follows that the rank of a tangentially degenerate hypersurface U^n is determined by the rank of the matrix (ν_{ij}) in terms of which the 1-forms ω_i^{n+1} and ω_0^i are expressed. But by (19) and (21) the dimension of the submanifold V described by the point A_0 on the hyperquadric Q^n is also equal to the rank of the matrix (ν_{ij}). Thus we have proved the following result:

THEOREM 2. *A lightlike hypersurface of the de Sitter space S_1^{n+1} is a ruled tangentially degenerate hypersurface whose rank is equal to the dimension of the submanifold V described by the point A_0 on the hyperquadric Q^n.*

Denote the rank of the tensor ν_{ij} and of the hypersurface U^n by r. In this and next sections we will assume that $r = n - 1$, and the case $r < n - 1$ will be considered in the last section of the paper.

For $r = n - 1$, the hypersurface U^n carries an $(n - 1)$-parameter family of isotropic rectilinear generators $l = A_n A_0$ along which the tangent hyperplane $T_x(U^n)$ is fixed. From the point of view of physics, the isotropic rectilinear generators of a lightlike hypersurface U^n are trajectories of light impulses, and the hypersurface U^n itself represents a *light flux* in spacetime.

Since rank $(\nu_{ij}) = n - 1$, the submanifold V described by the point A_0 on the hyperquadric Q^n has dimension $n - 1$, that is, V is a hypersurface. We denote it by V^{n-1}. The tangent subspace $T_{A_0}(V^{n-1})$ to V^{n-1} is determined by the points $A_0, A_1, \ldots, A_{n-1}$. Since

$$(A_n, A_i) = 0,$$

this tangent subspace is polar-conjugate to the rectilinear generator $A_0 A_n$ of the lightlike hypersurface U^n.

The submanifold V^{n-1} of the hyperquadric Q^n is the image of a hypersurface of the conformal space C^n under the Darboux mapping. We will denote this hypersurface also by V^{n-1}. In the space C^n, the hypersurface V^{n-1} is defined by Equation (18) which by (5) is equivalent to Equation (17) defining a lightlike hypersurface U^n in the space S_1^{n+1}. To the rectilinear generator $A_n A_0$ of the hypersurface U^n there corresponds a parabolic pencil of hyperspheres $A_n + sA_0$ tangent to the hypersurface V^{n-1} (see [7, p. 40]). Thus, the following theorem is valid:

THEOREM 3. *There exists a one-to-one correspondence between the set of hypersurfaces of the conformal space C^n and the set of lightlike hypersurfaces of the maximal rank $r = n - 1$ of the de Sitter space S_1^{n+1}. To pencils of tangent hyperspheres of the hypersurface V^{n-1} there correspond isotropic rectilinear generators of the lightlike hypersurface U^n.*

Note that for lightlike hypersurfaces of the four-dimensional Minkowski space M^4 the result similar to the result of Theorem 2 was obtained in [12].

3. The Fundamental Forms and Connections on a Lightlike Hypersurface of the de Sitter Space

The first fundamental form of a lightlike hypersurface U^n of the space S_1^{n+1} is a metric quadratic form. It is defined by the scalar square of the differential $\mathrm{d}x$ of a point of this hypersurface. Since we have $x = A_n$, by (16) and (2) this scalar square has the form

$$(\mathrm{d}A_n, \mathrm{d}A_n) = g_{ij}\omega_n^i\omega_n^j = g \tag{22}$$

and is a positive semidefinite differential quadratic form of signature $(n-1, 0)$. It follows that the system of equations $\omega_n^i = 0$ defines on the hypersurface U^n a fibration of isotropic lines which, as we showed in Section 2, coincide with rectilinear generators of this hypersurface.

The second fundamental form of a lightlike hypersurface U^n determines its deviation from the tangent hyperplane η. To find this quadratic form, we compute the part of the second differential of the point A_n which does not belong to the tangent hyperplane $\eta = A_0 \wedge A_1 \wedge \cdots \wedge A_n$:

$$d^2 A_n \equiv \omega_n^i \omega_i^{n+1} A_{n+1} \pmod{\eta}.$$

This implies that the second fundamental form can be written as

$$b = \omega_n^i \omega_i^{n+1} = \nu_{ij}\omega_n^i\omega_n^j, \tag{23}$$

where we used expression (20) for the form ω_i^{n+1}. Since we assumed that rank $(\nu_{ij}) = n-1$, the rank of the quadratic form (23) as well as the rank of the form g is equal to $n-1$. The nullspace of this quadratic form (see [13, p. 53]) is again determined by the system of equations $\omega_n^i = 0$ and coincides with the isotropic direction on the hypersurface U^n. The reduction of the rank of the quadratic form b is connected with the tangential degeneracy of the hypersurface U^n. The latter was noted in Theorem 2.

On a hypersurface V^{n-1} of the conformal space C^n that corresponds to a lightlike hypersurface $U^n \subset S_1^{n+1}$, the quadratic forms (22) and (23) define the net of curvature lines, that is, an orthogonal and conjugate net.

To find the connection forms of the hypersurface U^n, we find exterior derivatives of its basis forms ω_n^0 and ω_n^i:

$$\begin{cases} \mathrm{d}\omega_n^0 = \omega_n^0 \wedge \omega_0^0 + \omega_n^i \wedge \omega_i^0, \\ \mathrm{d}\omega_n^i = \omega_n^0 \wedge \omega_0^i + \omega_n^j \wedge \omega_j^i. \end{cases} \tag{24}$$

This implies that the matrix 1-form

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_i^0 \\ \omega_0^i & \omega_j^i \end{pmatrix} \tag{25}$$

defines a torsion-free connection on the hypersurface U^n. To clarify the properties of this connection, we find its curvature forms. To this end, we substitute the null value $\omega_n^{n+1} = 0$ of the form ω_n^{n+1}, which by (17) defines U^n along with the frame subbundle associated with U^n in the space S_1^{n+1}, into expression (12) for the curvature forms of the de Sitter space S_1^{n+1}. As a result, we obtain

$$\begin{cases} \Omega_0^0 = \mathrm{d}\omega_0^0 - \omega_0^i \wedge \omega_i^0 = 0, \\ \Omega_0^i = \mathrm{d}\omega_0^i - \omega_0^0 \wedge \omega_0^i - \omega_0^j \wedge \omega_j^i = 0, \\ \Omega_i^0 = \mathrm{d}\omega_i^0 - \omega_i^0 \wedge \omega_0^0 - \omega_i^j \wedge \omega_j^0 = -g_{ij}\omega_n^j \wedge \omega_n^0, \\ \Omega_j^i = \mathrm{d}\omega_j^i - \omega_j^0 \wedge \omega_0^i - \omega_j^k \wedge \omega_k^i - \omega_j^{n+1} \wedge \omega_{n+1}^i = -g_{jk}\omega_n^k \wedge \omega_n^i. \end{cases} \tag{26}$$

In these formulas the forms ω_j^{n+1} and ω_0^i are expressed in terms of the basis forms ω_n^i, and the forms ω_0^j, ω_j^i, and ω_i^0 are fiber forms. If the principal parameters are fixed, then these fiber forms are invariant forms of the group G of admissible transformations of frames associated with a point $x = A_n$ of the hypersurface U^n, and the connection defined by the form (25) is a G-connection.

To assign an affine connection on the hypersurface U^n, it is necessary to make a reduction of the family of frames in such a way that the forms ω_i^0 become principal. Denote by δ the symbol of differentiation with respect to the fiber parameters, that is, for a fixed point $x = A_n$ of the hypersurface U^n, and by π_η^ξ the values of the 1-forms ω_η^ξ for a fixed point $x = A_n$, that is, $\pi_\eta^\xi = \omega_\eta^\xi(\delta)$. Then we obtain

$$\pi_n^0 = 0, \qquad \pi_n^i = 0, \qquad \pi_i^n = 0, \qquad \pi_i^{n+1} = 0.$$

It follows that

$$\delta A_i = \pi_i^0 A_0 + \pi_i^j A_j. \tag{27}$$

The points A_0 and A_i determine the tangent subspace to the submanifold V^{n-1} described by the point A_0 on the hyperquadric Q^n. If we fix an $(n-2)$-dimensional subspace ζ not containing the point A_0 in this tangent subspace and place the points A_i into ζ, then we obtain π_i^0. This means that the forms ω_i^0 become principal, that is,

$$\omega_i^0 = \mu_{ij}\omega_n^j + \mu_i\omega_n^0, \tag{28}$$

and as a result, an affine connection arises on the hypersurface U^n.

We will call the subspace $\zeta \subset T_{A_0}(V^{n-1})$ the *normalizing subspace* of the lightlike hypersurface U^n. We have proved the following result:

THEOREM 4. *If in every tangent subspace $T_{A_0}(V^{n-1})$ of the submanifold V^{n-1} associated with a lightlike hypersurface U^n, $V^{n-1} \subset Q^n$, a normalizing $(n-2)$-dimensional subspace ζ not containing the point A_0 is assigned, then there arises a torsion-free affine connection on U^n.*

The last statement of Theorem 4 follows from the first two equations of (26).

By (28), the last equation of (26) can be written in the form

$$\begin{aligned}\widetilde{\Omega}^i_j &= \mathrm{d}\omega^i_j - \omega^k_j \wedge \omega^i_k \\ &= g^{im}(-g_{jk}g_{ml} + \mu_{jk}\nu_{ml} + \nu_{jk}\mu_{ml})\omega^k_n \wedge \omega^l_n + \\ &\quad + g^{im}(\mu_j \nu_{ml} - \mu_m \nu_{jl})\omega^0_n \wedge \omega^l_n.\end{aligned} \tag{29}$$

From the first three equations of (26) and Equation (29) we can find the torsion tensor of the affine connection indicated in Theorem 4:

$$\begin{aligned} R^0_{0uv} &= 0, \qquad R^i_{0uv} = 0, \qquad R^0_{ij0} = -R^0_{i0j} = -\frac{1}{2}g^{ij}, \\ R^i_{jkl} &= \frac{1}{2}g^{im}(g_{jl}g_{mk} - g_{jk}g_{ml} + \mu_{jk}\nu_{ml} - \mu_{jl}\nu_{mk} + \\ &\quad + \nu_{jk}\mu_{ml} - \nu_{jl}\mu_{mk}), \\ R^i_{j0l} &= -R^i_{jl0} = \frac{1}{2}g^{im}(\mu_j\nu_{ml} - \mu_m\nu_{jl}). \end{aligned} \tag{30}$$

The constructed above fibration of normalizing subspaces ζ defines a distribution Δ of $(n-1)$-dimensional elements on a lightlike hypersurface U^n. In fact, the point $x = A_n$ of the hypersurface U^n along with the subspace

$$\zeta = A_1 \wedge \cdots \wedge A_{n-1}$$

define the $(n-1)$-dimensional subspace which is complementary to the straight line $A_n A_0$ and lies in the tangent subspace η of the hypersurface U^n. Following the book [10], we will call this subspace the *screen*, and the distribution Δ the *screen distribution*. Since at the point x the screen is determined by the subspace $A_n \wedge A_1 \wedge \cdots \wedge A_{n-1}$, the differential equations of the screen distribution has the form

$$\omega^0_n = 0. \tag{31}$$

But by (28)

$$\mathrm{d}\omega^0_n = \omega^i_n \wedge \left(\mu_{ij}\omega^j_n + \mu_i \omega^0_n\right).$$

Hence, the screen distribution is integrable if and only if the tensor μ_{ij} is symmetric. Thus we arrived at the following result:

THEOREM 5. *The fibration of normalizing subspaces ζ defines a screen distribution Δ of $(n-1)$-dimensional elements on a lightlike hypersurface U^n. This distribution is integrable if and only if the tensor μ_{ij} defined by Equation* (28) *is symmetric.*

Note that the configurations similar to that described in Theorem 5 occurred in the works of the Moscow geometers published in the 1950s. They were called the *one-side stratifiable pairs of ruled surfaces* (see [11, §30] or [6, p. 187]).

4. An Invariant Normalization of Lightlike Hypersurfaces of the de Sitter Space

In [1] (see also [7, Ch. 2]) an invariant normalization of a hypersurfaces V^{n-1} of the conformal space C^n was constructed. By Theorem 3, this normalization can be interpreted in terms of the geometry of the de Sitter space S_1^{n+1}.

Taking exterior derivative of Equations (18) defining the hypersurface V^{n-1} in the conformal space C^n, we obtain

$$\omega_i^n \wedge \omega_0^i = 0,$$

from which by linear independence of the 1-forms ω_0^i on V^{n-1} and Cartan's lemma we find that

$$\omega_i^n = \lambda_{ij}\omega_0^j, \quad \lambda_{ij} = \lambda_{ji}. \tag{32}$$

Here, and in what follows, we retain the notations used in the study of the geometry of hypersurfaces of the conformal space C^n in the book [7].

It is not difficult to find relations between the coefficients ν_{ij} in formulas (20) and λ_{ij} in formulas (32). Substituting the values of the forms ω_i^n and ω_0^j from (5) into (32), we find that

$$-g_{ij}\omega_n^j = \lambda_{ij}g^{jk}\omega_k^{n+1}.$$

Solving these equations for ω_k^{n+1}, we obtain

$$\omega_i^{n+1} = -g_{ik}\widetilde{\lambda}^{kl}g_{lj}\omega_n^j,$$

where $(\widetilde{\lambda}^{kl})$ is the inverse matrix of the matrix (λ_{ij}). Comparing these equations with Equations (20), we obtain

$$\nu_{ij} = -g_{ik}\widetilde{\lambda}^{kl}g_{lj}. \tag{33}$$

Of course, in this computation we assumed that the matrix (λ_{ij}) is nondegenerate.

Let us clarify the geometric meaning of the vanishing of $\det(\lambda_{ij})$. To this end, we make an admissible transformation of the moving frame associated with a point of a lightlike hypersurface U^n by setting

$$\widehat{A}_n = A_n + sA_0. \tag{34}$$

The point $\widehat{A}_n$ as the point A_n lies on the rectilinear generator A_nA_0. Differentiating this point and applying formulas (16) and (19), we obtain

$$\mathrm{d}\widehat{A}_n = \left(\mathrm{d}s + s\omega_0^0 + \omega_n^0\right)A_0 + \left(\omega_n^i + s\omega_0^i\right)A_i. \tag{35}$$

It follows that in the new frame the form ω_n^i becomes

$$\widehat{\omega}_n^i = \omega_n^i + s\omega_0^i.$$

By (5) and (32), it follows that

$$\widehat{\omega}_n^i = -g^{ik}(\lambda_{kj} - s g_{kj})\omega_0^j.$$

This implies that in the new frame the quantities λ_{ij} become

$$\widehat{\lambda}_{ij} = \lambda_{ij} - s g_{ij}. \tag{36}$$

Consider also the matrix $(\widehat{\lambda}_j^i) = (g^{ik}\widehat{\lambda}_{kj})$. Since g_{ij} is a nondegenerate tensor, the matrices $(\widehat{\lambda}_j^i)$ and $(\widehat{\lambda}_{ij})$ have the same rank $\rho \leqslant n-1$.

From Equation (35) it follows that

$$\mathrm{d}\widehat{A}_n = \big(\mathrm{d}s + s\omega_0^0 + \omega_n^0\big)A_0 - \widehat{\lambda}_j^i A_i \omega_0^j.$$

Hence, the tangent subspace to the hypersurface U^n at the point $\widehat{A}_n$ is determined by the points $\widehat{A}_n$, A_0, and $\widehat{\lambda}_j^i A_i$. At the points, at which the rank ρ of the matrix $(\widehat{\lambda}_j^i)$ is equal to $n-1$, $\rho = n-1$, the tangent subspace to the hypersurface U^n has dimension n, and such points are *regular points* of the hypersurface. The points, at which the rank ρ of the matrix $(\widehat{\lambda}_j^i)$ is reduced, are *singular points* of the hypersurface U^n. The coordinates of singular points are defined by the condition $\det(\widehat{\lambda}_j^i) = 0$ which by (36) is equivalent to the equation

$$\det(\lambda_{ij} - s g_{ij}) = 0, \tag{37}$$

the *characteristic equation* of the matrix (λ_{ij}) with respect to the tensor g_{ij}. The degree of this equation is equal to $n-1$.

In particular, if A_n is a regular point of the hypersurface U^n, then the matrix (λ_{ij}) is nondegenerate, and Equation (33) holds. On the other hand, if A_n is a singular point of U^n, then Equation (33) is meaningless.

Since the matrix (λ_{ij}) is symmetric and the matrix (g_{ij}) defines a positive definite form of rank $n-1$, Equation (37) has $n-1$ real roots if each root is counted as many times as its multiplicity. Thus on a rectilinear generator $A_n A_0$ of a lightlike hypersurface U^n there are $n-1$ real singular points.

By Vieta's theorem, the sum of the roots of Equation (37) is equal to the coefficient in s^{n-2}, and this coefficient is $\lambda_{ij} g^{ij}$. Consider the quantity

$$\lambda = \frac{1}{n-1}\lambda_{ij} g^{ij}, \tag{38}$$

which is the arithmetic mean of the roots of Equation (37). This quantity λ allows us to construct new quantities

$$a_{ij} = \lambda_{ij} - \lambda g_{ij}. \tag{39}$$

It is easy to check that the quantities a_{ij} do not depend on the location of the point A_n on the straight line $A_n A_0$, that is, a_{ij} is invariant with respect to the transformation of the moving frame defined by Equation (34). Thus, the quantities

a_{ij} form a tensor on the hypersurface U^n defined in its second-order neighborhood. This tensor satisfies the condition

$$a_{ij} g^{ij} = 0, \tag{40}$$

that is, it is apolar to the tensor g_{ij}.

On the straight line $A_n A_0$ we consider a point

$$C = A_n + \lambda A_0. \tag{41}$$

It is not difficult to check that this point remains also fixed when the point A_n moves along the straight line $A_n A_0$. Since λ is the arithmetic mean of the roots of Equation (37) defining singular points on the straight line $A_n A_0$, the point C is the *harmonic pole* (see [9]) of the point A_0 with respect to these singular points. In particular, for $n = 3$, the point C is the fourth harmonic point to the point A_0 with respect to two singular points of the rectilinear generator $A_3 A_0$ of the lightlike hypersurface U^3 of the de Sitter space S_1^4.

In the conformal theory of hypersurfaces, to the point C there corresponds a hypersphere which is tangent to the hypersurface at the point A_0. This hypersphere is called the *central tangent hypersphere* (see [7, pp. 40–41]). Since

$$(\mathrm{d}^2 A_0, C) = a_{ij} \omega_0^i \omega_0^j, \tag{42}$$

the cone

$$a_{ij} \omega_0^i \omega_0^j = 0$$

with vertex at the point A_0 belonging to the tangent subspace $T_{A_0}(V^{n-1})$ contains the directions along which the central hypersphere has a second-order tangency with the hypersurface V^{n-1} at the point A_0. From the apolarity condition (39) it follows that it is possible to inscribe an orthogonal $(n-1)$-hedron with vertex at A_0 into the cone defined by Equation (42) (see [6, pp. 214–216]).

Now we can construct an invariant normalization of a lightlike hypersurface U^n of the de Sitter space S_1^{n+1}. To this end, first we repeat some computations from Ch. 2 of [7].

Taking exterior derivatives of Equations (32) and applying Cartan's lemma, we obtain the equations

$$\nabla \lambda_{ij} + \lambda_{ij} \omega_0^0 + g_{ij} \omega_n^0 = \lambda_{ijk} \omega_0^k, \tag{43}$$

where

$$\nabla \lambda_{ij} = \mathrm{d}\lambda_{ij} - \lambda_{ik} \omega_j^k - \lambda_{kj} \omega_i^k,$$

and the quantities λ_{ijk} are symmetric with respect to all three indices. Equations (43) confirm one more time that the quantities λ_{ij} do not form a tensor and depend on a location of the point A_n on the straight line $A_n A_0$. This dependence is described by a closed form relation (36). From formulas (43) it follows that the quantity λ defined by Equations (38) satisfy the differential equation

$$\mathrm{d}\lambda + \lambda\omega_0^0 + \omega_n^0 = \lambda_k \omega_0^k, \tag{44}$$

where

$$\lambda_k = \frac{1}{n-1} g^{ij} \lambda_{ijk}$$

(see formulas (2.1.35) and (2.1.36) in the book [7]).

The point C lying on the rectilinear generator $A_n A_0$ of the hypersurface U^n describes a submanifold $W \subset U^n$ when $A_n A_0$ moves. Let us find the tangent subspace to U^n at the point C. Differentiating Equation (40) and applying formulas (16) and (19), we obtain

$$\mathrm{d}C = \big(\mathrm{d}\lambda + \lambda\omega_0^0 + \omega_n^0\big) A_0 + \big(\omega_n^i + \lambda\omega_0^i\big) A_i.$$

By (5), (32), (39), and (44), it follows that

$$\mathrm{d}C = \big(\lambda_i A_0 - g^{jk} a_{ki} A_j\big)\omega_0^i. \tag{45}$$

Define the affinor

$$a_j^i = g^{ik} a_{kj}, \tag{46}$$

whose rank coincides with the rank of the tensor a_{ij}. Then Equation (45) takes the form

$$\mathrm{d}C = \big(\lambda_i A_0 - a_i^j A_j\big)\omega_0^i.$$

The points

$$C_i = \lambda_i A_0 - a_i^j A_j \tag{47}$$

together with the point C define the tangent subspace to the submanifold W described by the point C on the hypersurface U^n.

If the point C is a regular point of the rectilinear generator $A_n A_0$ of the hypersurface U^n, then the rank of the tensor a_{ij} defined by Equations (39) as well as the rank of the affinor a_j^i is equal to $n-1$. As a result, the points C_i are linearly independent and together with the point C define the $(n-1)$-dimensional tangent subspace $T_C(W)$, and the submanifold W itself has dimension $n-1$, $\dim W = n-1$.

The points C_i also belong to the tangent subspace $T_{A_0}(V^{n-1})$ and define the $(n-2)$-dimensional subspace $\zeta = T_{A_0}(V^{n-1}) \cap T_C(W)$ in it. This subspace is a normalizing subspace. Since such a normalizing subspace is defined in each tangent subspace $T_{A_0}(V^{n-1})$ of the hypersurface $V^{n-1} \subset Q^n$, there arises the fibration of these subspaces which by Theorem 4 defines an invariant affine connection on the lightlike hypersurface U^n. Thus we have proved the following result:

THEOREM 6. *If the tensor* a_{ij} *defined by formula* (39) *on a lightlike hypersurface* $U^n \subset S_1^{n+1}$ *is nondegenerate, then it is possible to construct the invariant normalization of* U^n *by means of the* $(n-2)$*-dimensional subspaces*

$$\zeta = C_1 \wedge C_2 \wedge \cdots \wedge C_{n-1}.$$

This normalization induces on U^n *an invariant screen distribution and an invariant affine connection intrinsically connected with the geometry of this hypersurface.*

Theorem 5 implies that the invariant normalization we have constructed defines on U^n an invariant screen distribution Δ which is also intrinsically connected with the geometry of the hypersurface U^n; here $\Delta_x = x \wedge \xi$, $x \in A_n A_0$.

Note that for the hypersurface V^{n-1} of the conformal space C^n a similar invariant normalization was constructed as far back as 1952 (see [1] and also [7, Ch. 2]). In the present paper we gave a new geometric meaning of this invariant normalization.

5. Singular Points of Lightlike Hypersurfaces of the de Sitter Space

As we indicated in Section 4, the points

$$z = A_n + sA_0 \tag{48}$$

of the rectilinear generator $A_n A_0$ of the lightlike hypersurface U^n are singular if their nonhomogeneous coordinate s satisfies the equation

$$\det(\lambda_{ij} - s g_{ij}) = 0. \tag{49}$$

We will investigate in more detail the structure of a lightlike hypersurface U^n in a neighborhood of its singular point.

Equation (49) is the characteristic equation of the matrix (λ_{ij}) with respect to the tensor (g_{ij}). The degree of this equation is $n-1$, and since the matrix (λ_{ij}) is symmetric and the matrix (g_{ij}) is also symmetric and positive definite, then according to the well-known result of linear algebra, all roots of this equation are real, and the matrices (λ_{ij}) and (g_{ij}) can be simultaneously reduced to a diagonal form.

Denote the roots of the characteristic equation by s_h, $h = 1, 2, \ldots, n-1$, and denote the corresponding singular points of the rectilinear generator $A_n A_0$ by

$$B_h = A_n + s_h A_0. \tag{50}$$

These singular points are called *foci* of the rectilinear generator $A_n A_0$ of a lightlike hypersurface U^n.

It is clear from (50) that the point A_0 is not a focus of the rectilinear generator $A_n A_0$. This is explained by the fact that by our assumption rank $(\nu_{ij}) = n-1$, and by (21), on the hyperquadric Q^n the point A_0 describes a hypersurface V^{n-1} which is transversal to the straight lines $A_0 A_n$.

In the conformal theory of hypersurfaces, to the singular points B_h there correspond the tangent hyperspheres defining the principal directions at a point A_0 of the hypersurface V^{n-1} of the conformal space C^n (see [7, p. 55]).

We will construct a classification of singular points of a lightlike hypersurface U^n of the space S_1^{n+1}. We will use some computations that we made while constructing a classification of canal hypersurfaces in [8].

Suppose first that $B_1 = A_n + s_1 A_0$ be a singular point defined by a simple root s_1 of characteristic Equation (49), $s_1 \neq s_h$, $h = 2, \dots, n-1$. For this singular point we have

$$\mathrm{d}B_1 = \left(\mathrm{d}s_1 + s_1\omega_0^0 + \omega_n^0\right)A_0 - \widehat{\lambda}_j^i \omega_0^j A_i, \tag{51}$$

where

$$\widehat{\lambda}_j^i = g^{ik}(\lambda_{kj} - s_1 g_{kj}) \tag{52}$$

is a degenerate symmetric affinor having a single null eigenvalue. The matrix of this affinor can be reduced to a quasidiagonal form

$$(\widehat{\lambda}_j^i) = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\lambda}_q^p \end{pmatrix}, \tag{53}$$

where $p, q = 2, \dots, n-1$, and $(\widehat{\lambda}_q^p)$ is a nondegenerate symmetric affinor. The matrices (g_{ij}) and $(\lambda_{ij} - s_1 g_{ij})$ are reduced to the forms

$$\begin{pmatrix} 1 & 0 \\ 0 & g_{pq} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\lambda}_{pq} \end{pmatrix},$$

where $(\widehat{\lambda}_{pq}) = (\lambda_{pq} - s_1 g_{pq})$ is a nondegenerate symmetric matrix.

Since the point B_1 is defined invariantly on the generator $A_n A_0$, then it will be fixed if $\omega_0^i = 0$. Thus it follows from (51) that

$$\mathrm{d}s_1 + s_1\omega_0^0 + \omega_n^0 = s_{1i}\omega^i; \tag{54}$$

here and in what follows $\omega^i = \omega_0^i$. By (53) and (54) relation (51) takes the form

$$\mathrm{d}B_1 = s_{11}\omega^1 A_0 + \left(s_{1p}A_0 - \widehat{\lambda}_p^q A_q\right)\omega^p. \tag{55}$$

Here the points $C_p = s_{1p}A_0 - \lambda_p^q A_q$ are linearly independent and belong to the tangent subspace $T_{A_0}(V^{n-1})$.

Consider the submanifold $\mathcal{F}_1$ described by the singular point B_1 in the space S_1^{n+1}. This submanifold is called the *focal manifold* of the hypersurface U^n. Relation (55) shows that two cases are possible:

(1) $s_{11} \neq 0$. In this case the submanifold $\mathcal{F}_1$ is of dimension $n-1$, and its tangent subspace at the point B_1 is determined by the points B_1, A_0, and C_p. This subspace contains the straight line $A_n A_0$, intersects the hyperquadric Q^n, and thus it, as well as the submanifold $\mathcal{F}_1$ itself, is timelike. For $\omega^p = 0$, the point B_1 describes a curve γ on the submanifold $\mathcal{F}_1$ which is tangent to the straight line $B_1 A_0$ coinciding with the generator $A_n A_0$ of the hypersurface U^n. The curve γ is an isotropic curve of the de Sitter space S_1^{n+1}. Thus, on $\mathcal{F}_1$ there arises a fiber bundle of focal lines. The hypersurface U^n is foliated into an $(n-2)$-parameter family of torses for which these lines are edges of regressions. The points B_1 are singular points of a kind which is called a *fold*.

If the characteristic Equation (49) has distinct roots, then an isotropic rectilinear generator l of a lightlike hypersurface U^n carries $n-1$ distinct foci $B_h, h = 1, \ldots, n-1$. If for each of these foci the condition of type $s_{11} \neq 0$ holds, then each of them describes a focal submanifold $\mathcal{F}_h$, carrying a conjugate net. Curves of one family of this net are tangent to the straight lines l, and this family is isotropic. On the hypersurface V^{n-1} of the space $C^n = Q^n$ described by the point A_0, to these conjugate nets there corresponds the net of curvature lines.

(2) $s_{11} = 0$. In this case relation (55) takes the form

$$\mathrm{d}B_1 = \left(s_{1p} A_0 - \widehat{\lambda}_p^q A_q\right)\omega^p, \tag{56}$$

and the focal submanifold $\mathcal{F}_1$ is of dimension $n-2$. Its tangent subspace at the point B_1 is determined by the points B_1 and C_p. An arbitrary point z of this subspace can be written in the form

$$z = z^n B_1 + z^p C_p = z^n (A_n + s_1 A_0) + z^p \left(s_{1p} A_0 - \widehat{\lambda}_p^q A_q\right).$$

Substituting the coordinates of this point into relation (3), we find that

$$(z, z) = g_{rs} \widehat{\lambda}_p^r \widehat{\lambda}_q^s z^p z^q + (z^n)^2 > 0.$$

It follows that the tangent subspace $T_{B_1}(F_1)$ does not have common points with the hyperquadric Q^n, that is, it is spacelike. Since this takes place for any point $B_1 \in \mathcal{F}_1$, the focal submanifold $\mathcal{F}_1$ is spacelike.

For $\omega^p = 0$, the point B_1 is fixed. The subspace $T_{B_1}(\mathcal{F}_1)$ will be fixed too. On the hyperquadric Q^n, the point A_0 describes a curve q which is polar-conjugate to $T_{B_1}(\mathcal{F}_1)$. Since $\dim T_{B_1}(\mathcal{F}_1) = n-2$, the curve q is a conic, along which the two-dimensional plane polar-conjugate to the subspace $T_{B_1}(\mathcal{F}_1)$ with respect to the hyperquadric Q^n, intersects Q^n. Thus, for $\omega^p = 0$, the rectilinear generator $A_n A_0$ of the hypersurface U^n describes a two-dimensional second-order cone with vertex at the point B_1 and the directrix q. Hence, in the case under consideration a lightlike hypersurface U^n is foliated into an $(n-2)$-parameter family of second-order cones whose vertices describe the $(n-2)$-dimensional focal submanifold $\mathcal{F}_1$, and the points B_1 are *conic* singular points of the hypersurface U^n.

The hypersurface V^{n-1} of the conformal space C^n corresponding to such a lightlike hypersurface U^n is a canal hypersurface which envelops an $(n-2)$-parameter family of hyperspheres. Such a hypersurface carries a family of cyclic generators which depends on the same number of parameters. Such hypersurfaces were investigated in detail in [8].

Further let B_1 be a singular point of multiplicity m, where $m \geqslant 2$, of a rectilinear generator $A_n A_0$ of a lightlike hypersurface U^n of the space S_1^{n+1} defined by an m-multiple root of characteristic Equation (49). We will assume that

$$s_1 = s_2 = \cdots = s_m := s_0, \quad s_0 \neq s_p, \tag{57}$$

and also assume that $a, b, c = 1, \dots, m$ and $p, q, r = m+1, \dots, n-1$. Then the matrices (g_{ij}) and (λ_{ij}) can be simultaneously reduced to quasidiagonal forms

$$\begin{pmatrix} g_{ab} & 0 \\ 0 & g_{pq} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_0 g_{ab} & 0 \\ 0 & \lambda_{pq} \end{pmatrix}.$$

We also construct the matrix $(\widehat{\lambda}_{ij}) = (\lambda_{ij} - s_0 g_{ij})$. Then

$$(\widehat{\lambda}_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\lambda}_{pq} \end{pmatrix}, \tag{58}$$

where $\widehat{\lambda}_{pq} = \lambda_{pq} - s_0 g_{pq}$ is a nondegenerate matrix of order $n - m - 1$.

By relations (58) and formulas (5) and (32) we have

$$\omega_a^n - s_0 \omega_a^{n+1} = 0, \tag{59}$$

$$\omega_p^n - s_0 \omega_p^{n+1} = \widehat{\lambda}_{pq} \omega^q. \tag{60}$$

Taking exterior derivative of Equation (59) and applying relation (60), we find that

$$\widehat{\lambda}_{pq} \omega_a^p \wedge \omega^q + g_{ab} \omega^b \wedge \left(\mathrm{d}s_0 + s_0 \omega_0^0 + \omega_n^0\right) = 0. \tag{61}$$

It follows that the 1-form $\mathrm{d}s_0 + s_0\omega_0^0 + \omega_n^0$ can be expressed in terms of the basis forms. We write these expressions in the form

$$\mathrm{d}s_0 + s_0 \omega_0^0 + \omega_n^0 = s_{0c} \omega^c + s_{0q} \omega^q. \tag{62}$$

Substituting this decomposition into Equation (61), we find that

$$\left(\widehat{\lambda}_{pq} \omega_a^p + g_{ab} s_{0q} \omega^b\right) \wedge \omega^q + g_{ab} s_{0c} \omega^b \wedge \omega^c = 0. \tag{63}$$

The terms in the left hand side of (63) do not have similar terms. Hence, both terms are equal to 0. Equating to 0 the coefficients of the summands of the second term, we find that

$$g_{ab} s_{0c} = g_{ac} s_{0b}. \tag{64}$$

Contracting this equation with the matrix (g^{ab}) which is the inverse matrix of the matrix (g_{ab}), we obtain

$$m s_{0c} = s_{0c}.$$

Since $m \geqslant 2$, it follows that

$$s_{0c} = 0,$$

and relation (62) takes the form

$$\mathrm{d}s_0 + s_0\omega_0^0 + \omega_n^0 = s_{0p}\omega^p. \tag{65}$$

For the singular point of multiplicity m of the generator A_nA_0 in question Equation (51) can be written in the form

$$\mathrm{d}B_1 = \big(\mathrm{d}s_0 + s_0\omega_0^0 + \omega_n^0\big)A_0 - \widehat{\lambda}_q^p\omega_0^q A_p.$$

Substituting decomposition (65) in the last equation, we find that

$$\mathrm{d}B_1 = \big(s_{0p}A_0 - \widehat{\lambda}_p^q A_q\big)\omega_0^p. \tag{66}$$

This relation is similar to Equation (56) with the only difference that in (56) we had $p, q = 2, \dots, n-1$, and in (66) we have $p, q = m+1, \dots, n-1$. Thus, the point B_1 describes now a spacelike focal manifold $\mathcal{F}_1$ of dimension $n - m - 1$. For $\omega_0^p = 0$, the point B_1 is fixed, and the point A_0 describes an m-dimensional submanifold on the hyperquadric Q^n which is a cross-section of Q^n by an $(m + 1)$-dimensional subspace that is polar-conjugate to the $(n - m - 1)$-dimensional subspace tangent to the submanifold $\mathcal{F}_1$.

The point B_1 is a conic singular point of multiplicity m of a lightlike hypersurface U^n, and this hypersurface is foliated into an $(n - m - 1)$-parameter family of $(m + 1)$-dimensional second-order cones circumscribed about the hyperquadric Q^n. The hypersurface V^{n-1} of the conformal space C^n that corresponds to such a hypersurface U^n is an m-canal hypersurface (i.e., the envelope of an $(n - m - 1)$-parameter family of hyperspheres), and it carries an m-dimensional spherical generators.

Note also an extreme case when the rectilinear generator A_nA_0 of a lightlike hypersurface U^n carries a single singular point of multiplicity $n - 1$. As follows from our consideration of the cases $m \geqslant 2$, this singular point is fixed, and the hypersurface U^n becomes a second-order hypercone with vertex at this singular point which is circumscribed about the hyperquadric Q^n. This hypercone is the isotropic cone of the space S_1^{n+1}. The hypersurface V^{n-1} of the conformal space C^n that corresponds to such a hypersurface U^n is a hypersphere of the space C^n.

The following theorem combines the results of this section:

THEOREM 7. *A lightlike hypersurface U^n of maximal rank $r = n - 1$ of the de Sitter space S_1^{n+1} possesses $n - 1$ real singular points on each of its rectilinear generators if each of these singular points is counted as many times as its multiplicity. The simple singular points can be of two kinds: a fold and conic. In the first case the hypersurface U^n is foliated into an $(n - 2)$-parameter family of torses, and in the second case it is foliated into an $(n - 2)$-parameter family of second-order cones. The vertices of these cones describe the $(n-2)$-dimensional spacelike* *submanifolds in the space* S_1^{n+1}. *All multiple singular points of a hypersurface* U^n *are conic. If a rectilinear generator of a hypersurface* U^n *carries a singular point of multiplicity* m, $2 \leqslant m \leqslant n-1$, *then the hypersurface* U^n *is foliated into an* $(n-m-1)$*-parameter family of* $(m+1)$*-dimensional second-order cones. The vertices of these cones describe the* $(n-m-1)$*-dimensional spacelike submanifold in the space* S_1^{n+1}. *The hypersurface* V^{n-1} *of the conformal space* C^n *corresponding to a lightlike hypersurface* U^n *with singular points of multiplicity* m *is a canal hypersurface which envelops an* $(n-m-1)$*-parameter family of hyperspheres and has* m*-dimensional spherical generators.*

Since lightlike hypersurfaces U^n of the de Sitter space S_1^{n+1} represent a light flux (see Section 2), its focal submanifolds have the following physical meaning. If one of them is a lighting submanifold, then others will be manifolds of concentration of a light flux. Intensity of concentration depends on multiplicity of a focus describing this submanifold.

In the extreme case when an isotropic rectilinear generator $l = A_n A_0$ of a hypersurface U^n carries one $(n-1)$-multiple focus, the hypersurface U^n degenerates into the light cone generated by a point source of light. This cone represents a radiating light flux.

If each isotropic generator $l \subset U^n$ carries two foci B_1 and B_2 of multiplicities m_1 and m_2, $m_1 + m_2 = n-1$, $m_1 > 1$, $m_2 > 1$, then these foci describe spacelike submanifolds $\mathcal{F}_1$ and $\mathcal{F}_2$ of dimension $n - m_1 - 1$ and $n - m_2 - 1$, respectively. If one of these submanifolds is a lighting submanifold, then on the second one a light flux is concentrated.

6. Lightlike Hypersurfaces of Reduced Rank

As we proved in Section 2, lightlike hypersurfaces of the de Sitter space S_1^{n+1} are ruled tangentially degenerate hypersurfaces. However in all preceding sections starting from Section 3 we assumed that the rank of these hypersurfaces is maximal, that is, it is equal to $n-1$. In this section we consider lightlike hypersurfaces of reduced rank $r < n-1$.

We proved in Section 2 that the rank of a lightlike hypersurface U^n coincides with the rank of the matrix (ν_{ij}) defined by Equation (20) as well as with the dimension of the submanifold V described by the point A_0 on the Darboux hyperquadric Q^n. As a result, to a lightlike hypersurface U^n of rank r there corresponds an r-dimensional submanifold $V = V^r$ in the conformal space C^n.

The symmetric matrices (g_{ij}) and (ν_{ij}) first of which is nondegenerate and positive definite and second is of rank r, can be simultaneously reduced to quasidiagonal forms

$$(g_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{pq} \end{pmatrix} \quad \text{and} \quad (\nu_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \nu_{pq} \end{pmatrix}, \tag{67}$$

where $a, b = 1, \dots, m$; $p, q, s = m+1, \dots, n-1$, $\nu_{pq} = \nu_{qp}$, and $\det(\nu_{pq}) \neq 0$. This implies that formulas (21) take the form

$$\omega_0^a = 0, \qquad \omega_0^p = g^{ps} \nu_{sq} \omega_n^q. \tag{68}$$

The last equation in (68) show that the 1-forms ω_0^p are linearly independent: they are basis forms on the submanifold $V = V^r$ described by the point A_0 on the hyperquadric Q^n, on the lightlike hypersurface U^n of rank r, and also on a frame bundle associated with this hypersurface. The 1-forms occurring in Equations (4) as linear combinations of the basis forms ω_0^p are called principal forms, and the 1-forms that are not expressed in terms of the basis forms are fiber forms on the above mentioned frame bundle.

By (5) the second group of Equations (68) is equivalent to the system of equations

$$\omega_p^n = \lambda_{pq}^n \omega_0^q, \tag{69}$$

where $\lambda_{pq}^n = -g_{ps} \widetilde{\nu}^{st} g_{ts}$, $(\widetilde{\nu}^{st})$ is the inverse matrix of the matrix (ν_{pq}), $\lambda_{pq}^n = \lambda_{qp}^n$, and $\det(\lambda_{pq}^n) \neq 0$. Note that we can also obtain Equations (69) by differentiation of Equation (18) which holds on the hypersurface U^n.

Taking exterior derivatives of the first group of Equations (68), we find that

$$\omega_0^p \wedge \omega_p^a = 0.$$

Applying Cartan's lemma to this system, we find that

$$\omega_p^a = \lambda_{pq}^a \omega_0^q, \quad \lambda_{pq}^a = \lambda_{qp}^a. \tag{70}$$

Note also that Equations (5) and (67) imply that

$$g_{pq} \omega_a^q + g_{ab} \omega_p^b = 0.$$

By (70), it follows from the last equation that

$$\omega_a^p = -g_{ab} g^{pq} \lambda_{qs}^b \omega_0^s. \tag{71}$$

Note also that the quantities λ_{pq}^a and λ_{pq}^n are determined in a second-order neighborhood of a rectilinear generator $l = A_0 A_n$ of the hypersurface U^n.

Let us prove that in our frame an $(m+1)$-dimensional span L of the points A_0, A_a, and A_n is a plane generator of the lightlike hypersurface U^n. In fact, it follows from Equations (4) that in the case in question we have

$$\begin{cases} \mathrm{d}A_0 = \omega_0^0 A_0 + \omega_0^p A_p, \\ \mathrm{d}A_a = \omega_a^0 A_0 + \omega_a^b A_b + \omega_a^p A_p + \omega_a^n A_n, \\ \mathrm{d}A_n = \omega_n^0 A_0 + \omega_n^a A_a + \omega_n^p A_p. \end{cases} \tag{72}$$

If we fix the principal parameters in Equations (72) (i.e., if we assume that $\omega_0^p = 0$), we obtain

$$\begin{cases} \delta A_0 = \pi_0^0 A_0, \\ \delta A_a = \pi_a^0 A_0 + \pi_a^b A_b + \pi_a^n A_n, \\ \delta A_n = \pi_n^0 A_0 + \pi_n^a A_a. \end{cases} \tag{73}$$

In the last equations δ is the symbol of differentiation with respect to the fiber parameters (i.e., for $\omega_0^p = 0$), and $\pi_\eta^\xi = \omega_\eta^\xi(\delta)$.

Equations (73) show that for $\omega_0^p = 0$, the point A_n of the hypersurface U_n moves in an $(m+1)$-dimensional domain belonging to the subspace $L = A_0 \wedge A_1 \wedge \cdots \wedge A_m \wedge A_n$ of the same dimension. Let us assume that the entire subspace L belongs to the hypersurface U^n, and that the point $A_n \in L$ moves freely in L. The subspace L is tangent to the hyperquadric Q^n at the point $A_0 \in V^r$, and thus L is lightlike. Since the point A_0 describes an r-dimensional submanifold, the family of subspaces L depends on r parameters. Hence, $U^n = f(M^r \times L)$, where f is a differentiable map $f\colon\ M^r \times L \to P^{n+1}$.

Equations (72) and (73) show that the basis 1-forms of a lightlike hypersurface U^n are divided into two classes: ω_n^p and ω_n^a. The forms ω_n^p are connected with the displacement of the lightlike $(m+1)$-plane L in the space S_1^{n+1}, and the forms ω_n^a are connected with the displacement of the straight line $A_n A_0$ in this $(m+1)$-plane. Since (73) implies that for $\omega_n^p = 0$ the point A_0 remains fixed, the rectilinear generator $A_n A_0$ describes an m-dimensional bundle of straight lines with its center at the point A_0, and this bundle belongs to the fixed $(m+1)$-dimensional subspace L passing through this point.

Further consider an arbitrary point

$$z = z^0 A_0 + z^a A_a + z^n A_n \tag{74}$$

of the generator L of the lightlike hypersurface U^n. From formulas (72) it follows that the differential of any such point belongs to one and the same n-dimensional subspace $A_0 \wedge \cdots \wedge A_n$ tangent to the hypersurface U^n at the original point A_n. The latter means that the tangent subspace to the hypersurface U^n is not changed when the point z moves along the lightlike generator L of the hypersurface U^n. Thus, hypersurface U^n is a tangentially degenerate hypersurface of rank r.

As a result, we arrive at the following theorem making Theorem 2 more precise:

THEOREM 8. *If the rank of the tensor ν_{ij} defined by relation* (20) *is equal to r, $r < n-1$, then a lightlike hypersurface U^n of the de Sitter space S_1^{n+1} is a ruled tangentially degenerate hypersurface of rank r with $(m+1)$-dimensional lightlike generators, $m = n - r - 1$, along which the tangent hyperplanes of U^n are constant. The points of tangency of lightlike generators with the hyperquadric Q^n form an r-dimensional submanifold V^r on Q^n.*

The last fact mentioned in Theorem 8 can be also treated in terms of quadratic hyperbands (see [6, p. 256]). By Theorem 8, the hypersurface U^n is the envelope of an r-parameter family of hyperplanes η tangent to the hyperquadric Q^n at the points of an r-dimensional smooth submanifold V^r belonging to this hyperquadric. But this coincides precisely with the definition of the quadratic hyperband. Thus, Theorem 8 can be complemented as follows:

THEOREM 9. *A lightlike hypersurface U^n of rank r in the de Sitter space S_1^{n+1} is an r-dimensional quadratic hyperband with the support submanifold V^r belonging to the Darboux hyperquadric Q^n.*

Note also an extreme case when the rank of a lightlike hypersurface U^n is equal to 0. Then we have

$$\nu_{ij} = 0, \qquad \omega_0^i = 0.$$

The point A_0 is fixed on the hyperquadric Q^n, and the point A_n moves freely in the hyperplane η tangent to the hyperquadric Q^n at the point A_0. The lightlike hypersurface U^n degenerates into the hyperplane η tangent to the hyperquadric Q^n at the point A_0, and the quadratic hyperband associated with U^n is reduced to a 0-pair consisting of the point A_0 and the hyperplane η.

Let us also find singular points on a rectilinear generator L of a lightlike hypersurface U^n of rank r of the de Sitter space S_1^{n+1}. To this end, we write the differential of a point $z \in L$ defined by Equation (74). We will be interested only in that part of this differential which does not belong to the generator L. By (72), we obtain

$$\mathrm{d}z \equiv \left(z^0\omega_0^p + z^a\omega_a^p + z^n\omega_n^p\right)A_p \quad (\mathrm{mod}\ L).$$

By (69), (70) and (71), we find from the last relation that

$$\mathrm{d}z \equiv N_q^p(z)\omega_0^q A_p \quad (\mathrm{mod}\ L),$$

where

$$N_q^p(z) = \delta_q^p z^0 - g_{ab}g^{ps}\lambda_{sq}^b z^a - g^{ps}\lambda_{sq}^n z^n. \tag{75}$$

At singular points of a generator L the dimension of the tangent subspace $T_x(U^n)$ to the hypersurface U^n is reduced. By (75), this is equivalent to the reduction of the rank of the matrix $N_q^p(z)$. Thus, singular points of generator L can be found from the condition

$$\det N_q^p(z) = 0, \tag{76}$$

which defines an algebraic focal submanifold $\mathcal{F}$ of dimension m and order r in the $(m+1)$-dimensional plane generator L. The left-hand side of Equation (76) is the Jacobian of the map $f\colon\ M^r \times L \to P^{n+1}$ indicated above, and the focal submanifold $\mathcal{F}$ is the locus of singular points of this map that are located in the plane generator L of the hypersurface U^n.

If the rank of a lightlike hypersurface U^n is maximal, that is, it is equal to $r = n - 1$, then its determinant manifold $\mathcal{F}$ is a set of singular points of its rectilinear generator $A_n A_0$ determined by Equation (50). On the other hand, if $r < n - 1$, then singular points of the straight lines $A_n A_0$ lying in the generator L are also determined by Equation (50), and they are the intersection points of these straight lines and the manifold $\mathcal{F}$.

In a plane generator L of the lightlike hypersurface U^n, let us find an equation of the harmonic polar of the point A_0 with respect to the algebraic focal submanifold $\mathcal{F}$. Let us assume that the coordinates z^a and z^n in Equation (74) are fixed, and the coordinate z^0 is variable. Then the point z describes a straight line $l = A_0 \wedge (z^a A_a + z^n A_n)$. The intersection point of this line l with the focal submanifold $\mathcal{F}$ is determined by Equation (76) in which the quantities z^a and z^n are fixed, and z^0 is variable. Equation (76) is of degree r with respect to z^0 and defines r focal (singular) points on the straight line l if each of these points is counted as many times as its multiplicity. By the Vieta theorem, the coefficient in $(z^0)^{r-1}$ in Equation (76) is equal to the sum of roots of this equation. Thus,

$$\frac{1}{r} \sum_{p=m+1}^{n-1} z_p^0 = z^a g_{ab} \lambda^b + z^n \lambda^n,$$

where z_p^0 are roots of Equation (76), and the quantities λ^b and λ^n are defined by the formulas

$$\lambda^a = \frac{1}{r} g^{pq} \lambda_{pq}^a, \qquad \lambda^n = \frac{1}{r} g^{pq} \lambda_{pq}^n. \tag{77}$$

Thus, the harmonic pole of the point A_0 with respect to the foci of the straight line l has the form

$$C = \big(g_{ab} \lambda^a z^b + \lambda^n z^n\big) A_0 + z^a A_a + z^n A_n.$$

The locus of these poles on a plane generator L of the hypersurface U^n is defined by the equation

$$z^0 - g_{ab} \lambda^a z^b - \lambda^n z^n = 0, \tag{78}$$

whose left-hand side is the trace of the matrix $N_q^p(z)$. Equation (78) defines a subspace of dimension m on the $(m + 1)$-dimensional generator l of the hypersurface U^n. This subspace is the harmonic polar of the point A_0 with respect to the algebraic focal submanifold $\mathcal{F}$.

For construction of screen distribution on a lightlike hypersurface U^n of rank $r < n - 1$ we will need differential prolongations of Pfaffian equations (69) and (70). Taking exterior derivatives of these equations, we find that

$$\big(\nabla \lambda_{pq}^a + \lambda_{pq}^a \omega_0^0 + \lambda_{pq}^n \omega_n^0 + g_{pq} \omega_{n+1}^a\big) \wedge \omega_0^q = 0, \tag{79}$$

$$\big(\nabla \lambda_{pq}^n + \lambda_{pq}^n \omega_0^0 + \lambda_{pq}^a \omega_a^n + g_{pq} \omega_{n+1}^n\big) \wedge \omega_0^q = 0, \tag{80}$$

where

$$\nabla\lambda^a_{pq} = \mathrm{d}\lambda^a_{pq} - \lambda^a_{sq}\omega^s_p - \lambda^a_{ps}\omega^s_q + \lambda^b_{pq}\omega^a_b,$$
$$\nabla\lambda^n_{pq} = \mathrm{d}\lambda^n_{pq} - \lambda^n_{sq}\omega^s_p - \lambda^n_{ps}\omega^s_q.$$

Applying Cartan's lemma to Equations (79) and (80) and fixing the principal parameters (i.e., setting $\omega^p_0 = 0$), we find that

$$\begin{cases} \nabla_\delta\lambda^a_{pq} + \lambda^a_{pq}\pi^0_0 + \lambda^a_{pq}\pi^0_n + g_{pq}\pi^a_{n+1} = 0, \\ \nabla_\delta\lambda^n_{pq} + \lambda^n_{pq}\pi^0_0 + \lambda^a_{pq}\pi^n_a + g_{pq}\pi^n_{n+1} = 0. \end{cases} \tag{81}$$

Note also that by the last equation of Equations (5), the tensor g_{pq} defined by the first group of relations (67) satisfies the equations

$$\nabla_\delta g_{pq} = 0. \tag{82}$$

Equations (81) and (82) prove that neither quantities λ^a_{pq} nor quantities λ^n_{pq} form a geometric object, but jointly, the quantities λ^a_{pq}, λ^n_{pq}, and the tensor g_{pq} form a linear geometric object.

By Equations (81) and (82), the quantities λ^a and λ^n defined by formulas (77) satisfy the equations

$$\begin{cases} \nabla_\delta\lambda^a + \lambda^a\pi^0_0 + \lambda^n\pi^a_n + \pi^a_{n+1} = 0, \\ \nabla_\delta\lambda^n + \lambda^n\pi^0_0 + \lambda^a\pi^n_a + \pi^n_{n+1} = 0. \end{cases} \tag{83}$$

It follows that jointly, the quantities λ^a and λ^n form a geometric object which is associated with a second-order differential neighborhood of a plane generator L of the hypersurface U^n.

Next we construct the quantities

$$a^a_{pq} = \lambda^a_{pq} - \lambda^a g_{pq}, \qquad a^n_{pq} = \lambda^n_{pq} - \lambda^n g_{pq}. \tag{84}$$

By (80), (81), and (82), they satisfy the equations

$$\begin{cases} \nabla_\delta a^a_{pq} + a^a_{pq}\pi^0_0 + a^n_{pq}\pi^0_n = 0, \\ \nabla_\delta a^n_{pq} + a^n_{pq}\pi^0_0 + a^a_{pq}\pi^n_a = 0. \end{cases} \tag{85}$$

These equations prove that jointly the quantities a^a_{pq} and a^n_{pq} form a tensor with respect to the admissible transformations in a frame bundle associated with the hypersurface U^n. Let us assume that the indices α and β take $m+1$ values $1, \ldots, m, n$. Then the equations, which the tensor $a^\alpha_{pq} = \{a^a_{pq}, a^n_{pq}\}$ satisfies, can be written in the form

$$\nabla_\delta a^\alpha_{pq} + a^\alpha_{pq}\pi^0_0 = 0, \tag{86}$$

where $\nabla_\delta a^\alpha_{pq} = \delta a^\alpha_{pq} - a^\alpha_{sq}\pi^s_p - a^\alpha_{ps}\pi^s_q + a^\beta_{pq}\pi^\alpha_\beta$. It follows from Equations (84) that this tensor satisfies the apolarity condition

$$a^\alpha_{pq} g^{pq} = 0. \tag{87}$$

Consider the straight lines A_0A_α connecting the point A_0 with the points A_α, $\alpha = 1, \dots, m, n$. Their parametric equations can be written in the form

$$\widetilde{A}_\alpha = A_\alpha + x_\alpha A_0.$$

Let us find the points of intersection of these straight lines with the harmonic polar K of the point A_0 with respect to the focal submanifold $\mathcal{F}$. Substituting coordinates of these points into Equation (78), we find that

$$x_\alpha = \lambda_\alpha, \quad \alpha = 1, \dots, m, n,$$

where $\lambda_a = g_{ab}\lambda^b$ and $\lambda_n = \lambda^n$.

The points $C_\alpha = A_\alpha + \lambda_\alpha A_0$ lying in the subspace K can be taken as the vertices of a reduced frame associated with the hypersurface U^n and defined in a second-order differential neighborhood of the plane generator L of this hypersurface. If we consider our hypersurface with respect to this reduced frame, then we have

$$\lambda^\alpha = 0, \qquad \lambda^\alpha_{pq} = a^\alpha_{pq}. \tag{88}$$

It follows from Equation (83) that the 1-forms ω^α_{n+1} become principal forms:

$$\omega^\alpha_{n+1} = b^\alpha_p \omega^p. \tag{89}$$

With respect to the new frame, Equations (69) and (70) take the form

$$\omega^\alpha_p = a^\alpha_{pq}\omega^q, \quad a^\alpha_{pq} = a^\alpha_{qp}. \tag{90}$$

Consider the rectangular matrix $A = (a^\alpha_{pq})$ in which α is the row number, and the pair $(p, q) = (q, p)$ is the column number. The matrix A has $m + 1$ rows and $\frac{1}{2}r(r+1)$ columns. But by (87), not more than $\frac{1}{2}r(r+1)-1$ columns of the matrix A are linearly independent. Suppose that $\operatorname{rank} A = \rho$, $\rho \leqslant \min\{m+1, \frac{1}{2}r(r+1)-1\}$. Construct the following tensors:

$$a^{\alpha\beta} = g^{pq}g^{st}a^\alpha_{ps}a^\beta_{qt} \quad \text{and} \quad a^\alpha_\beta = g_{\beta\gamma}a^{\gamma\alpha}. \tag{91}$$

It is not difficult to prove that the rank of each of these tensors is equal to the rank of the matrix A, $\operatorname{rank}(a^{\alpha\beta}) = \operatorname{rank}(a^\alpha_\beta) = \rho$.

Construct the quantity

$$a = a^{\alpha_1}_{[\alpha_1} a^{\alpha_2}_{\alpha_2} \dots a^{\alpha_\rho}_{\alpha_\rho]},$$

which is equal to the sum of the diagonal minors of order ρ of the matrix (a^α_β). Since the rank of this matrix is equal to ρ, then if $\rho \geqslant 1$, the quantity a is different from 0, $a \neq 0$.

From Equations (86) it follows that the tensor a^α_β satisfies the equations

$$\nabla_\delta a^\alpha_\beta + 2a^\alpha_\beta \pi^0_0 = 0.$$

Applying the formula for differentiation of determinants, we find from the last equation that the quantity a satisfies the equation

$$\delta a + 2\rho a \pi^0_0 = 0, \tag{92}$$

i.e., a is a relative invariant of weight -2ρ.

Equation (92) is written for fixed principal parameters, i.e., under the condition $\omega^p_0 = 0$. If these parameters are variable, then it follows from Equation (92) that

$$\frac{\mathrm{d}a}{2\rho a} + \omega^0_0 = \mu_p \omega^p_0. \tag{93}$$

Taking the exterior derivative of the last equation, we find that

$$\left(\mathrm{d}\mu_p - \mu_q \omega^q_p + \omega_p \omega^0_0 + \omega^0_p\right) \wedge \omega^p_0 = 0. \tag{94}$$

This implies that the quantities μ_p form a geometric object. For $\omega^p_0 = 0$, this object satisfies the equations

$$\nabla_\delta \mu_p + \mu_p \pi^0_0 + \pi^0_p = 0. \tag{95}$$

It follows from Equation (93) that the geometric object μ_p is defined in a third-order differential neighborhood of the plane generator L of the hypersurface U^n.

Consider the subspace $T_{A_0}(V^r) = A_0 \wedge A_{m+1} \wedge \cdots \wedge A_{n-1}$ tangent to the submanifold V^r described by the point A_0 on the hyperquadric Q^n. This subspace belongs to the tangent hyperplane η to the lightlike hypersurface U^n and is orthogonal to its plane generator $L = A_0 \wedge A_1 \wedge \cdots \wedge A_m \wedge A_n$. The geometric object μ_p allows us to construct a normalizing subspace ζ of dimension $r-1$ in $T_{A_0}(V^r)$. To this end, consider the points

$$\widetilde{A}_p = A_p + x_p A_0.$$

Differentiating these points, applying Equations (5), (69), and (70), and assuming that the principal parameters are fixed, i.e., $\omega^p_0 = 0$, we find that

$$\delta \widetilde{A}_p = \left(\nabla_\delta x_p + x_p \pi^0_0 + \pi^0_p\right) A_0 + \pi^q_p \widetilde{A}_q.$$

This implies that the subspace spanned by the points $\widetilde{A}_p$ is invariant if and only if the quantities x_p satisfy the differential equations

$$\nabla_\delta x_p + x_p \pi^0_0 + \pi^0_p = 0.$$

Comparing these equations with Equations (95), we see that they are satisfied if we take $x_p = \mu_p$. Thus, the points

$$C_p = A_p + \mu_p A_0$$

determine an invariant normalizing subspace $\zeta = C_{m+1} \wedge \cdots \wedge C_{n-1}$.

Suppose that x is an arbitrary point of the generator L of the hypersurface U^n. This point and the subspace ζ define an r-plane $\Delta_x = x \wedge \zeta$. Such r-planes are defined for all points $x \in U^n$ and form an r-dimensional *screen distribution* Δ on U^n which is complementary to the generators L of U^n. Since the geometric object μ_p is defined in a third-order neighborhood, then the screen distribution is defined in the same neighborhood. Thus the following theorem holds.

THEOREM 10. *If the rank of the matrix $A = (a^\alpha_{pq})$ is different from* 0*, then in a third-order neighborhood of a plane generator L of a lightlike hypersurface U^n of rank $r < n-1$, there is defined an invariant r-dimensional screen distribution Δ.*

If we place the vertices A_p of our frame into the points C_p, then we obtain $\mu_p = 0$. This and Equation (94) imply that

$$\omega^0_p = c_{pq}\omega^q_0, \quad c_{pq} = c_{qp}. \tag{96}$$

The points C_p and the normalizing subspace $\zeta = C_{m+1} \wedge \cdots \wedge C_{n-1}$ are defined in a third-order neighborhood of a plane generator L, and the quantities c_{pq} are defined in a fourth-order neighborhood.

Let us prove that the fibration of normalizing subspaces we have constructed determines an affine connection on a hypersurface U^n which can be considered as an r-parameter fibration of its $(m+1)$-dimensional plane generators L. In fact, the basic forms of this fibration are the 1-forms ω^p_0. Taking exterior derivatives of these forms, we find that

$$\mathrm{d}\omega^p_0 = \omega^q_0 \wedge \theta^p_q, \tag{97}$$

where $\theta^p_q = \omega^p_q - \delta^p_q\omega^0_0$. Taking exterior derivatives of the forms θ^p_q and taking into account that by (96) $\mathrm{d}\omega^0_0 = 0$, we obtain

$$\mathrm{d}\theta^p_q - \theta^s_q \wedge \theta^p_s = R^p_{qst}\omega^s_0 \wedge \omega^t_0, \tag{98}$$

where

$$R^p_{qst} = -g_{\alpha\beta}g^{pu}a^\alpha_{q[s}a^\beta_{t]u} + c_{q[s}\delta^p_{t]} + g_{q[s}g^{pu}c_{t]u}, \tag{99}$$

and

$$g_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the following theorem is valid.

THEOREM 11. *An invariant screen distribution induces a torsion-free affine connection on the fibration of plane generators of a hypersurface U^n. The curvature tensor of this connection is determined by Equation* (99)*, and its tensor Ricci is symmetric.*

The last statement of Theorem 11 can be proved by a direct calculation. In fact, contracting Equation (99) with respect to the indices p and t, we find that

$$R_{pq} = R^s_{pqs} = \frac{1}{2}\big(g_{\alpha\beta} g^{st} a^\alpha_{ps} a^\beta_{qt} + (r-2)c_{pq} + g_{pq} g^{st} c_{st}\big),$$

It is easy to see that this tensor is symmetric, $R_{pq} = R_{qp}$.

We did not consider yet only the case when the rank of the matrix $A = (a^\alpha_{pq})$ is equal to 0, $\rho = 0$. In this case the matrix A is the null-matrix, and the construction of an invariant screen distribution is impossible. Let us clarify the geometric structure of the hypersurface U^n in this case.

If this is the case, formulas (84) imply that

$$\lambda^a_{pq} = \lambda^a g_{pq}, \qquad \lambda^n_{pq} = \lambda^n g_{pq}. \tag{100}$$

Thus, the Jacobi matrix (75) of the mapping $f\colon\ M^r \times L \to\ P^{n+1}$ takes the form

$$N^p_q(z) = \delta^p_q\big(z^0 - g_{ab}\lambda^b - \lambda^n\big),$$

and the equation of the focal submanifold $\mathcal{F}$ becomes

$$\det N^p_q(z) = \big(z^0 - g_{ab}\lambda^b z^a - z^n\big)^r = 0.$$

Thus, the focal submanifold $\mathcal{F}$ is an r-fold linear subspace

$$z^0 - g_{ab}\lambda^b z^a - z^n = 0 \tag{101}$$

of dimension m belonging to the $(m+1)$-dimensional plane generator L of the hypersurface U^n. It is possible to prove that if $r \geqslant 2$, then this subspace is fixed, and the hypersurface U^n is an n-dimensional cone with an $(m+1)$-dimensional plane generators and an m-dimensional vertex defined by Equation (101).

In this case the submanifold V^r, along which the hypersurface U^n is tangent to the hyperquadric Q^n, is an r-dimensional sphere S^r.

If $r = 1$, then the hypersurface U^n is an envelope of a one-parameter family of isotropic hyperplanes that are tangent to the hyperquadric Q^n at the points of an arbitrary curve γ. Finally if $r = 0$, then the hypersurface U^n is an isotropic hyperplane.

Note that the invariant normalization of a lightlike hypersurface U^n which we have constructed in Section 6 is a new geometric interpretation of a conformally invariant normalization of a submanifold V^r of a conformal space C^n which was constructed in [2] (see also [7, Ch. 3]).

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