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Quasi–Coisometric Realizations of Upper Triangular Matrices

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Abstract. In this paper we study analogs of de Branges–Rovnyak spaces and prove a realization theorem in the setting of upper triangular matrices

1 Introduction

Given a $\mathbb{C}^{p\times q}$–valued function S analytic and contractive in the open unit disk $\mathbb{D}$, the function

$$\frac{I_p - S(z)S(w)^*}{1 - zw^*} \tag{1.1}$$

is positive in $\mathbb{D}$. The associated reproducing kernel Hilbert space was introduced in [**10**], [**11**] by L. de Branges and J. Rovnyak. This space, which we will denote by $\mathcal{H}(S)$, plays an important role in various fields such as model theory of operators in Hilbert and Pontryagin spaces, interpolation theory and realization theory; see [**12**], [**1**]. The main property of the space $\mathcal{H}(S)$ of interest to us in this work is that $\mathcal{H}(S)$ is the state space of a coisometric realization of S. More precisely, one has

$$S(z) = D + zC(I_{\mathcal{H}(S)} - zA)^{-1}B, \tag{1.2}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{C}^q \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{C}^p \end{pmatrix} \tag{1.3}$$

is the backwards–shift realization defined by

$$(Au)(z) = \frac{u(z)-u(0)}{z} \tag{1.4}$$

$$Bf(z) = \frac{S(z)-S(0)}{z}f \tag{1.5}$$

$$Cu = u(0) \tag{1.6}$$

$$Df = S(0)f. \tag{1.7}$$

The matrix (1.3) is coisometric, and the realization (1.2) is closely outerconnected in the sense that $\cap_{n=0}^{\infty}\text{Ker } CA^n = \{0\}$. In general, it is not minimal but the closely outerconnectedness property insures uniqueness up to a similarity operator which moreover is unitary; see [**6**].

Analogs of $\mathcal{H}(S)$ spaces and of the associated *coisometric* backward shift representation appear in a surprising number of situations. For instance in the setting of upper triangular operators [**7**], lower triangular integral operators [**2**] and in the setting of compact real Riemann surfaces [**8**] and of the bidisk [**3**]. In [**1**, §2.5 p. 39] we showed how such spaces also appear in the setting of finite matrices, but no theory was elaborated. The purpose of this paper is to begin such a theory. An important feature here is that we lose the coisometricity property; see equation (5.9). The main results of the paper are the analogue of the backwards–shift realization (1.4)–(1.7), see Theorem 5.1, and the analogue of the realization formula (1.2), see formula (5.16).

2 Preliminaries

We denote by $\mathcal{X}_2^{n\times n}$ the Hilbert space of $\mathbb{C}^{n\times n}$ matrices, with the Hilbert–Schmidt inner product, i.e. for $F, G \in \mathbb{C}^{n\times n}$

$$\langle F, G\rangle_{\mathcal{X}_2} = \text{Tr}\, G^*F. \tag{2.1}$$

Recall that, for $A \in \mathbb{C}^{n\times n}$, one has

$$\langle AF, AF\rangle_{\mathcal{X}_2} \le \|A\|^2 \cdot \langle F, F\rangle_{\mathcal{X}_2}, \tag{2.2}$$

where $\|A\|$ denotes the operator norm, i.e. the largest eigenvalue of the positive matrix $(AA^*)^{\frac{1}{2}}$.

By $\mathcal{U}_2^{n\times n}$, $\mathcal{L}_2^{n\times n}$ and $\mathcal{D}_2^{n\times n}$ we denote the spaces of upper–triangular, lower–triangular and diagonal matrices, respectively endowed with the inner product (2.1). Sometimes we will write for simplicity $\mathcal{X}$, $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{D}$, especially when the Hilbert space structure is not present. We denote by I_n or I the $n\times n$ identity matrix. By Z we denote the $n\times n$ nilpotent matrix

$$Z = \begin{pmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 1 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

For any $W \in \mathcal{L}$ the matrix ZW^* is also nilpotent, and therefore

$$(I - ZW^*)^{-1} = \sum_{j=0}^{n-1}(ZW^*)^j.$$

Similarly, the matrix W^*Z is nilpotent and we have

$$(I - W^*Z)^{-1} = \sum_{j=0}^{n-1}(W^*Z)^j.$$

Lemma 2.1 *Let $F \in \mathcal{U}$. Then there exist uniquely defined diagonal matrices $F_{[j]}$, $j = 0, \dots, n-1$ such that $F = \sum_{j=0}^{n-1} Z^j F_{[j]}$ and*

$$F_{[j]} = Z^{*j}Z^jF_{[j]}, \quad j = 0, \dots n-1.$$

Similarly, there exist uniquely defined diagonal matrices $F_{\{j\}}$, $j = 0, \dots, n-1$ such that $F = \sum_{j=0}^{n-1} F_{\{j\}}Z^j$ and

$$F_{\{j\}} = F_{\{j\}}Z^jZ^{*j}, \quad j = 0, \dots n-1.$$

Proof We note that

$$Z^{*j}Z^j = \text{diag}\ (0, \dots 0, 1, \dots, 1)$$

where the first j elements of the diagonal are equal to 0. It is then easy to see that

$$F_{[j]} = \text{diag}\ (0, \dots, 0, f_{j+1,n}, \dots, f_{n-j,n})$$

where the first j elements of the diagonal are equal to 0. The other claims are proved similarly. □

Since

$$\mathcal{X}_2^{n\times n} = Z^*\mathcal{L}_2^{n\times n} \oplus \mathcal{D}_2^{n\times n} \oplus Z\mathcal{U}_2^{n\times n}$$

we have the natural projections $\mathbf{p}_- : \mathcal{X} \longrightarrow Z^*\mathcal{L}_2^{n\times n}$, $\mathbf{p}_0 : \mathcal{X}_2^{n\times n} \longrightarrow \mathcal{D}_2^{n\times n}$ and $\mathbf{p}_+ : \mathcal{X}_2^{n\times n} \longrightarrow Z\mathcal{U}_2^{n\times n}$, respectively. The projections $\mathbf{p}_- + \mathbf{p}_0$ and $\mathbf{p}_0 + \mathbf{p}_+$ from $\mathcal{X}_2^{n\times n}$ to $\mathcal{U}_2^{n\times n}$ and $\mathcal{L}_2^{n\times n}$ respectively, will be denoted by $\mathbf{p}$ and $\mathbf{q}$.

3 The point evaluations

In [**4**] and [**5**, Section 3] a point evaluation at a diagonal point is defined for an upper triangular operator. The analogue in the present setting is as follows: let $F \in \mathcal{U}$ and $W \in \mathcal{D}$. The (left sided) point evaluation of F at W is defined to be the diagonal matrix

$$F^\wedge(W) = \mathbf{p}_0\left((I - WZ^*)^{-1}F\right) = \sum_{j=0}^{n-1}(WZ^*)^jZ^jF_{[j]}. \tag{3.1}$$

Similarly, the (right sided) point evaluation of F at W is defined to be the diagonal matrix

$$F^\triangle(W) = \mathbf{p}_0\left(F(I - Z^*W)^{-1}\right) \tag{3.2}$$

The space $\mathcal{U}_2^{n\times n}$ is a reproducing kernel Hilbert space with reproducing kernel $(I - ZW^*)^{-1}$ in the following sense.

Theorem 3.1 *The linear span of the matrices of the form $(I - ZW^*)^{-1}E$ where $E \in \mathcal{D}_2^{n\times n}$ and $W \in \mathcal{D}^{n\times n}$, is equal to $\mathcal{U}_2^{n\times n}$. For such E, W and F we have*

$$\left\langle F, (I - ZW^*)^{-1}E\right\rangle_{\mathcal{U}_2^{n\times n}} = \text{Tr}\, E^*F^\wedge(W). \tag{3.3}$$

Proof The second claim follows from:

$$\begin{aligned}\left\langle F,(I-ZW^*)^{-1}E\right\rangle_{\mathcal{U}_2^{n\times n}} &= \operatorname{Tr} E^*(I-WZ^*)^{-1}F\\ &= \operatorname{Tr} E^*\mathbf{p}_0\left((I-WZ^*)^{-1}F\right)\\ &= \langle F^\wedge(W),E\rangle_{\mathcal{D}_2^{n\times n}}\\ &= \operatorname{Tr} E^*F^\wedge(W).\end{aligned}$$

If $\left\langle F,(I-ZW^*)^{-1}E\right\rangle_{\mathcal{U}_2^{n\times n}} = \langle F^\wedge(W),E\rangle_{\mathcal{D}_2^{n\times n}} = 0$ for all E and W as above, then $F^\wedge(W)=0$. Take $W=\lambda I$; hence,

$$F^\wedge(\lambda I)=\sum_{j=0}^{n-1}\lambda^j Z^{*j}Z^jF_{[j]}=\sum_{j=0}^{n-1}\lambda^jF_{[j]}=0$$

for any $\lambda\in\mathbb{C}$. Thus $F_{[0]}=F_{[1]}=\cdots=F_{[n-1]}=0$, which implies that $F=0$. □

We note that one has in a similar way

$$\left\langle F,E(I-W^*Z)^{-1}\right\rangle_{\mathcal{U}_2^{n\times n}}=\operatorname{Tr} F^\triangle(W)E^*.$$

Lemma 3.2 *Let $F,G\in\mathcal{U}$ and let $W\in\mathcal{D}$. Then,*

$$(FG)^\wedge(W)=(F^\wedge(W)G)^\wedge(W)$$

and

$$(FG)^\triangle(W)=(F(G^\triangle(W)))^\triangle(W).$$

Proof We have

$$FG=\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-1}Z^kF_{[k]}Z^\ell G_{[\ell]}$$

and thus

$$(FG)^\wedge(W)=\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-1}(WZ^*)^{k+\ell}Z^kF_{[k]}Z^\ell G_{[\ell]}.$$

On the other hand

$$F^\wedge(W)G=\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-1}(WZ^*)^kZ^kF_{[k]}Z^\ell G_{[\ell]}$$

and thus

$$(F^\wedge(W)G)^\wedge(W)=\sum_{k=1}^{n-1}\sum_{\ell=1}^{n-1}(WZ^*)^{k+\ell}Z^kF_{[k]}Z^\ell G_{[\ell]}.$$

The second equality is proved in the same way. □

When G is a diagonal operator we note that

$$(FG)^\wedge(W)=F^\wedge(W)G. \tag{3.4}$$

For similar results in the setting of upper triangular operators we refer to [**5**, Section 3].

4 The state space $\mathcal{H}(S)$

Let $\mathbf{H}_2^p$ denote the space of $\mathbb{C}^p$–valued functions with entries in the Hardy space of the unit disk $\mathbf{H}_2$ (see e.g. [**14**, pp. 320–323] for the definition of $\mathbf{H}_2$), and let S be as in the introduction. The operator $M_S^\ell : \mathbf{H}_2^q \to \mathbf{H}_2^p$ of multiplication by S on the left is a contraction and the operator range $\text{Ran}\left(\left(I - M_S^\ell M_S^{\ell *}\right)^{\frac{1}{2}}\right)$ endowed with the norm

$$\|\left(\left(I - M_S^\ell M_S^{\ell *}\right)^{\frac{1}{2}} u\|_{\text{Ran}\left(\left(I - M_S^\ell M_S^{\ell *}\right)^{\frac{1}{2}}\right)} = \|(I-\pi)u\|_{\mathbf{H}_2^p}$$

is the reproducing kernel Hilbert space $\mathcal{H}(S)$ with reproducing kernel (1.1); in this expression, π denotes the orthogonal projection onto the kernel of $\left(I - M_S^\ell M_S^{\ell *}\right)^{\frac{1}{2}}$. From [**9**, Theorem 3.9], one has the equivalent characterization:

$$\mathcal{H}(S) = \{f \in \mathbf{H}_2^p |\ \kappa(f) < \infty\}$$

where

$$\kappa(f) = \|f\|^2_{\mathcal{H}(S)} = \sup_{g \in \mathbf{H}_2^q} \left\{\|f + Sg\|^2_{\mathbf{H}_2^p} - \|g\|^2_{\mathbf{H}_2^q}\right\}.$$

We refer to [**13**, Theorem 4.1 p. 275] for more connections between operator ranges and the de Branges–Rovnyak spaces.

Motivated by the above discussion we define:

Definition 4.1 Let $S \in \mathbb{C}^{n\times n}$ be an upper triangular contractive matrix and let M_S^ℓ be the operator of multiplication from the left from $\mathcal{U}_2^{n\times n}$ into itself. The de Branges–Rovnyak space associated to S is defined as

$$\mathcal{H}(S) = \left\{F \in \mathcal{U}_2^{n\times n} |\ \kappa(F) < \infty\right\}$$

where

$$\kappa(F) = \|F\|^2_{\mathcal{H}(S)} = \sup_{G \in \mathcal{U}_2^{n\times n}} \left\{\|F + SG\|_{\mathcal{U}_2^{n\times n}} - \|G\|_{\mathcal{U}_2^{n\times n}}\right\}.$$

That the space $\mathcal{H}(S)$ is a Hilbert space follows from [**9**, Theorem 3.9] since the operator of multiplication by S on the left is a contraction from $\mathcal{U}_2^{n\times n}$ into itself. Still from that paper follows that $\mathcal{H}(S)$ is the range of the operator $\Gamma^{\frac{1}{2}}$, where now $\Gamma = I_{\mathcal{U}_2^{n\times n}} - M_S^\ell M_S^{\ell *}$, with M_S^ℓ the operator of multiplication from the left by the upper triangular contraction S from $\mathcal{U}_2^{n\times n}$ into itself. Γ is a positive operator and we have

$$\langle \Gamma^{\frac{1}{2}} U, \Gamma^{\frac{1}{2}} V\rangle_{\mathcal{H}(S)} = \langle U, V\rangle_{\mathcal{U}_2^{n\times n}}. \tag{4.1}$$

We set

$$\begin{aligned} K_W E &= (I - M_S^L (M_S^L)^*)((I - ZW^*)^{-1} E) && (4.2)\\ &= (I - SS^\wedge(W)^*)(I - ZW^*)^{-1} E. && (4.3)\end{aligned}$$

This is the analogue of the kernel (1.1). It follows easily from (4.1) that:

Proposition 4.2 *The space $\mathcal{H}(S)$ is a reproducing kernel Hilbert space with reproducing kernel K_W in the sense that*

$$\langle F, K_W E\rangle_{\mathcal{H}(S)} = \langle F^\wedge(W), E\rangle_{\mathcal{D}_2^{n\times n}} \tag{4.4}$$

for all $F \in \mathcal{H}(S)$ and $E, W \in \mathcal{D}$.

We conclude this section with

Proposition 4.3 *Let D be a diagonal operator. The operator M_D^r of multiplication by D on the right is everywhere defined and bounded from $\mathcal{H}(S)$ into itself with $\|M_D^r\| \le \|D\|$.*

Proof Let $F = \sum_{\ell=1}^m K_{W_\ell} E_\ell$ be an element of $\mathcal{H}(S)$. Then,

$$M_D^r F = \sum_{\ell=1}^m K_{W_\ell} E_\ell D \in \mathcal{H}(S).$$

Because of the finite dimensional hypothesis and since the $K_W E$ span all of $\mathcal{H}(S)$, this concludes that M_D^r is bounded. We now estimate the norm of M_D^r: we have

$$\|F\|^2_{\mathcal{H}(S)} = \text{Tr}\ \left(\sum E_j^* K_{W_\ell}^\wedge(W_j) E_\ell\right)$$

and

$$\begin{aligned}
\|M_D^r F\|^2_{\mathcal{H}(S)} &= \text{Tr}\ \left(\sum D^* E_j^* K_{W_\ell}^\wedge(W_j) E_\ell D\right)\\
&= \text{Tr}\ \left(\sum DD^* E_j^* K_{W_\ell}^\wedge(W_j) E_\ell\right)\\
&= \left\langle \sum E_j^* K_{W_\ell}^\wedge(W_j) E_\ell, DD^* \right\rangle_{\mathcal{X}_2^{n\times n}}\\
&\le \|D\| \cdot \|F\|_{\mathcal{H}(S)},
\end{aligned}$$

where we have used (3.4) and (2.2). □

5 The quasi–coisometric realization

The analogue of the backwards–shift realization (1.4)–(1.7) is given by:

Theorem 5.1 *The formulas*

$$\begin{aligned}
A(F) &= (F - F_{\{0\}})Z^* &(5.1)\\
B(E) &= (S - S_{\{0\}})EZ^* &(5.2)\\
C(F) &= F_{\{0\}} &(5.3)\\
D(E) &= S_{\{0\}}E &(5.4)
\end{aligned}$$

define an everywhere defined bounded operator

$$\begin{pmatrix} \mathcal{H}(S) \\ \mathcal{D}_2^{n\times n} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(S) \\ \mathcal{D}_2^{n\times n} \end{pmatrix}.$$

The adjoint colligation is given by

$$\begin{aligned}
A^*(K_W E) &= K_W E Z - S \cdot B^*(K_W E) &(5.5)\\
B^*(K_W E) &= Z^*\left(\left(S - S_{\{0\}}\right) Z^*\right)^\wedge (W) EZ &(5.6)\\
C^*(G) &= (I - SS^*_{\{0\}})G &(5.7)\\
D^*(E) &= S^*_{\{0\}} E &(5.8)
\end{aligned}$$

and is such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} M^r_{ZZ^*} & 0 \\ 0 & I \end{pmatrix} \qquad (5.9)$$

where $M^r_{ZZ^*}$ is the operator of multiplication from the right by the diagonal operator ZZ^*.

Proof We first show that A is everywhere defined:

$$\begin{aligned}
&\|(F - F_{\{0\}})Z^* + SG\|^2_{\mathcal{U}_2^{n\times n}} - \|G\|^2_{\mathcal{U}_2^{n\times n}} \\
&= \|F_{\{0\}}Z^*\|^2_{\mathcal{X}_2^{n\times n}} + \|FZ^* + SG\|^2_{\mathcal{X}_2^{n\times n}} \\
&\quad -2\text{Re } \langle F_{\{0\}}Z^*, FZ^* + SG\rangle_{\mathcal{X}_2^{n\times n}} - \|G\|^2_{\mathcal{U}_2^{n\times n}} \\
&= \|FZ^* + SG\|^2_{\mathcal{X}_2^{n\times n}} - \|F_{\{0\}}Z^*\|^2_{\mathcal{X}_2^{n\times n}} - \|G\|^2_{\mathcal{U}_2^{n\times n}},
\end{aligned}$$

since $\langle F_{\{0\}}Z^*, SG\rangle_{\mathcal{X}_2^{n\times n}} = 0$ and $\langle F_{\{0\}}Z^*, FZ^*\rangle_{\mathcal{X}_2^{n\times n}} = \langle F_{\{0\}}Z^*, F_{\{0\}}Z^*\rangle_{\mathcal{X}_2^{n\times n}}$. But we have

$$\begin{aligned}
\|FZ^* + SG\|^2 &= \|FZ^*\|^2 + \|SG\|^2 + 2\text{Re } \langle F, SGZ\rangle \\
&= \|F + SGZ\|^2 + \|FZ^*\|^2 - \|F\|^2 + \|SG\|^2 - \|SGZ\|^2
\end{aligned} \tag{5.10}$$

since

$$\langle FZ^*, SG\rangle = \text{Tr } Z^*G^*S^*F = \text{Tr } G^*S^*FZ^* = \langle F, SGZ\rangle.$$

Hence we can write (where, from now on, we denote the norm of matrices without the index $\mathcal{X}_2^{n\times n}$ to lighten the notation),

$$\begin{aligned}
&\|(F - F_{\{0\}})Z^* + SG\|^2_{\mathcal{U}_2^{n\times n}} - \|G\|^2_{\mathcal{U}_2^{n\times n}} \\
&= -\|F_{\{0\}}Z^*\|^2 + \|FZ^*\|^2 - \|F\|^2 + \|SG\|^2 - \|G\|^2 \\
&\quad +\|F + SGZ\|^2 - \|SGZ\|^2 \\
&\leq \|F\|^2_{\mathcal{H}(S)} + \|FZ^*\|^2 - \|F\|^2 - \|F_{\{0\}}Z^*\|^2 \\
&= \|F\|^2_{\mathcal{H}(S)} - \|F(I - ZZ^*)\|^2 - \|F_{\{0\}}Z^*\|^2 \\
&\leq \|F\|^2_{\mathcal{H}(S)} - \|F_{\{0\}}Z^*\|^2.
\end{aligned}$$

and it follows that A is bounded and in fact satisfies the inequality

$$\|AF\|^2_{\mathcal{H}(S)} \leq \|F\|^2_{\mathcal{H}(S)} - \|F_{\{0\}}Z^*\|^2. \tag{5.11}$$

We now turn to the operator B:

$$\begin{aligned}
&\|(S - S_{\{0\}})EZ^* + SG\|^2 - \|G\|^2 \\
&= \|S_{\{0\}}EZ^*\|^2 + \|SEZ^* + SG\|^2 \\
&\quad -2\text{Re } \langle S_{\{0\}}EZ^*, SEZ^* + SG\rangle - \|G\|^2 \\
&\leq -\|S_{\{0\}}EZ^*\|^2 + \|SEZ^*\|^2 + \|SG\|^2 - \|G\|^2 \\
&\leq \|SEZ^*\|^2 - \|S_{\{0\}}EZ^*\|^2 \\
&= \|(S - S_{\{0\}})EZ^*\|^2 \\
&\leq \|EZ^*\|^2 - \|S_{\{0\}}EZ^*\|^2 \\
&\leq \|E\|^2 - \|S_{\{0\}}EZ^*\|^2
\end{aligned}$$

and so B is bounded. The operators C and D are clearly bounded. We now compute the adjoint colligation: the computation of the adjoint of the operator B is as follows:

$$\begin{aligned}
\langle G, B^*(K_W E)\rangle_{\mathcal{D}_2^{n\times n}} &= \langle S - S_{\{0\}})GZ^*, K_W E\rangle_{\mathcal{H}(S)} \\
&= \langle ((S - S_{\{0\}})GZ^*)^\wedge(W), E\rangle_{\mathcal{D}_2^{n\times n}} \\
&= \text{Tr } E^*(I - WZ^*)^{-1}(S - S_{\{0\}})GZ^*
\end{aligned}$$

since

$$((S-S_{\{0\}})GZ^*)^\wedge(W)=\mathbf{p}_0\left((I-WZ^*)^{-1}(S-S_{\{0\}})GZ^*\right).$$

Since

$$\text{Tr } E^*(I-WZ^*)^{-1}(S-S_{\{0\}})GZ^*=\text{Tr } GZ^*E^*(I-WZ^*)^{-1}(S-S_{\{0\}})$$

we obtain that

$$B^*(K_WE)=\mathbf{p}_0\left(S^*(I-ZW^*)^{-1}EZ\right).$$

We now show that this expression coincides with (5.6): in view of the diagonal expansions

$$(I-ZW^*)^{-1}=\sum_{k=0}^{n-1}(ZW^*)^k,\qquad S^*=\sum_{k=o}^{n-1}Z^{*k}S^*_{\{k\}},$$

the main diagonal of $S^*(I-ZW^*)^{-1}EZ$ is equal to

$$\sum_{k=0}^{n-2}Z^{*(k+1)}S^*_{\{k+1\}}(ZW^*)^kEZ,$$

which can be rewritten as

$$Z^*\left(\sum_{k=0}^{n-2}Z^{*k}S^*_{\{k+1\}}(ZW^*)^k\right)EZ,$$

i.e. as (5.6).

We now compute C^*:

$$\begin{aligned}\langle K_WE,C^*(G)\rangle_{\mathcal{H}(S)}&=\langle (K_WE)_{\{0\}},G\rangle_{\mathcal{D}_2}\\&=\langle (K_WE)^\wedge(0),G\rangle_{\mathcal{D}_2}\\&=\langle K_WE,K_0G\rangle_{\mathcal{H}(S)},\end{aligned}$$

and hence $C^*(G)=K_0G=(I-SS^*_{\{0\}})(G)$.

The operator D^* is trivial to compute, and we compute A^*:

$$\begin{aligned}&\langle K_WG,A^*(K_WE)\rangle_{\mathcal{H}(S)}\\&=\langle A(K_WG),K_WE\rangle_{\mathcal{H}(S)}\\&=\langle (K_VG-(K_V)_{\{0\}}G)Z^*,K_WE\rangle_{\mathcal{H}(S)}\\&=\text{Tr } E^*(I_n-WZ^*)^{-1}\left(K_VG-(K_VG)_{\{0\}}\right)Z^*\\&=\text{Tr } E^*(I_n-WZ^*)^{-1}K_VGZ^*\\&=\text{Tr } E^*(I_n-WZ^*)^{-1}(I_n-SS^\wedge(V)^*)(I_n-ZV^*)^{-1}GZ^*\\&=\left(\text{Tr } G^*(I_n-VZ^*)^{-1}(I_n-S^\wedge(V)S^*)(I_n-ZW^*)^{-1}EZ\right)^*.\end{aligned}$$

But

$$\begin{aligned}&\text{Tr } G^*(I_n-VZ^*)^{-1}(I_n-S^\wedge(V)S^*)(I_n-ZW^*)^{-1}EZ\\&=\text{Tr } G^*(I_n-VZ^*)^{-1}(I_n-SS^\wedge(W)^*)(I_n-ZW^*)^{-1}EZ\qquad(5.12)\\&\quad+\text{Tr } G^*(I_n-VZ^*)^{-1}(SS^\wedge(W)^*-S^\wedge(V)S^*)(I_n-ZW^*)^{-1}EZ\qquad(5.13)\end{aligned}$$

The term (5.12) is equal to $\text{Tr } G^*(K_WEZ)^\wedge(V)$. To estimate the second term we rewrite it as

$$\begin{aligned}&\text{Tr } G^*(I_n-VZ^*)^{-1}(SS^\wedge(W)^*-S^\wedge(V)S^*)(I_n-ZW^*)^{-1}EZ\\&\quad=\text{Tr }\left(G^*(I_n-VZ^*)^{-1}SS^\wedge(W)^*(I_n-ZW^*)^{-1}E\right)\\&\qquad+\text{Tr }\left(G^*(I_n-VZ^*)^{-1}S\mathbf{p}S^*(I_n-ZW^*)^{-1}EZ\right),\end{aligned}$$

and show that

$$\mathbf{p}\left(S^*(I_n - ZW^*)^{-1}EZ\right) = S^\wedge(W)^*(I_n - ZW^*)^{-1}EZ + B^*(K_W E). \tag{5.14}$$

The formula for A^* then follows. To show (5.14), note that

$$\begin{aligned}
&\mathbf{p}\left(S^*(I_n - ZW^*)^{-1}EZ\right)\\
&= \mathbf{p}\left((S^* - S^\wedge(W)^* + S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right)\\
&= S^\wedge(W)^*(I_n - ZW^*)^{-1}EZ +\\
&\quad +\mathbf{p}\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right)\\
&= S^\wedge(W)^*(I_n - ZW^*)^{-1}EZ +\\
&\quad +\mathbf{p}_0\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right) +\\
&\quad +\mathbf{p}_+\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right)\\
&= S^\wedge(W)^*(I_n - ZW^*)^{-1}EZ +\\
&\quad +B^*(K_W EZ) + \mathbf{p}_+\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right)
\end{aligned}$$

and to show (5.14), it is enough to show that

$$\mathbf{p}_+\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right) = 0. \tag{5.15}$$

One has

$$\begin{aligned}
\mathbf{p}_+\left(S^*(I_n - ZW^*)^{-1}EZ\right) &= \mathbf{p}_+\left(\sum_{\ell=0}^{n-1}\sum_{k=0}^{n-1} Z^{*\ell} S^*_{\{\ell\}}(ZW^*)^k EZ\right)\\
&= \sum_{\ell=0}^{n-1}\sum_{k=\ell}^{n-1} Z^{*\ell} S^*_{\{\ell\}}(ZW^*)^k EZ,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{p}_+\left(S^\wedge(W)^*(I_n - ZW^*)^{-1}\right) &= \mathbf{p}_+\left(\sum_{\ell=0}^{n-1}\sum_{m=0}^{n-1} Z^{*\ell} S^*_{\{\ell\}}(ZW^*)^\ell (ZW^*)^m EZ\right)\\
&= \left(\sum_{\ell=0}^{n-1}\sum_{k=\ell}^{n-1+\ell} Z^{*\ell} S^*_{\{\ell\}}(ZW^*)^k EZ\right),
\end{aligned}$$

so that

$$\begin{aligned}
\mathbf{p}_+\left((S^* - S^\wedge(W)^*)(I_n - ZW^*)^{-1}EZ\right) &= \sum_{\ell=1}^{n-1}\sum_{k=n}^{n-1+\ell} Z^{*\ell} S^*_{\{\ell\}}(ZW^*)^k EZ\\
&= 0
\end{aligned}$$

since $(ZW^*)^k = 0$ for $k \geq n$. Hence (5.15) holds and so does (5.14).

We now check the "coisometry property" (5.9). First, we verify that $AC^* + BD^* = 0$. We have

$$\begin{aligned}
(AC^* + BD^*)(E) &= A\left(\left(I - SS^*_{\{0\}}\right)E\right) + B(S^*_{\{0\}}E)\\
&= \left(\left(I - SS^*_{\{0\}}\right) - \left(I - S_{\{0\}}S^*_{\{0\}}\right)\right)EZ^* + (S - S_{\{0\}})S^*_{\{0\}}E\\
&= 0.
\end{aligned}$$

Next, we verify that $CC^* + DD^* = I_{\mathcal{D}_2}$.

$$\begin{aligned}(CC^* + DD^*)(E) &= C((I - SS^*_{\{0\}})E) + D(S^*_{\{0\}}E)\\ &= (I - S_{\{0\}}S^*_{\{0\}})E + S_{\{0\}}S^*_{\{0\}}E\\ &= E.\end{aligned}$$

Finally,

$$\begin{aligned}AA^*(K_W E) &= A\left(K_W EZ - S \cdot B^*(K_W E)\right)\\ &= \left(K_W EZ - S \cdot B^*(K_W E) - (K_W EZ - S \cdot B^*(K_W E))_{\{0\}}\right) Z^*\\ &= \left(K_W EZ - S \cdot B^*(K_W E) + S_{\{0\}} \cdot B^*(K_W E)\right) Z^*\\ &= K_W EZZ^* - (S - S_{\{0\}})B^*(K_W E)Z^*\end{aligned}$$

and thus

$$(AA^* + BB^*)(K_W E) = K_W ZZ^*.$$

□

Theorem 5.2 *Let S be an upper triangular contraction, and let $\mathcal{H}(S)$ be the associated de Branges–Rovnyak space. Then, for $W \in \mathcal{D}$ of norm strictly less than 1,*

$$S^{\triangle}(W) = \left(D + CM^r_W(I_{\mathcal{H}(S)} - AM^r_W)^{-1}B\right)(I_n). \tag{5.16}$$

Proof We first prove that for $F \in \mathcal{H}(S)$, one has:

$$C(I_{\mathcal{H}(S)} - M^r_W A)^{-1}(F) = F^{\triangle}(W). \tag{5.17}$$

Indeed, let $(I_{\mathcal{H}(S)} - M^r_W A)^{-1}(F) = G$; then,

$$\begin{aligned}F &= G - M^r_W A(G)\\ &= G - A(G)W\\ &= G - (G - G_{\{0\}})Z^*W\\ &= G(I_n - Z^*W) + G_{\{0\}}Z^*W,\end{aligned}$$

so that

$$G = (F - G_{\{0\}}Z^*W)(I_n - Z^*W)^{-1}.$$

Applying $\mathbf{p}_0$ to this equation and using (3.2), we obtain $G_{\{0\}} = F^{\triangle}(W)$.

We obtain the realization formula (5.16) by using this formula with $F = B(I_n)W$: indeed, let $Y = M^r_W(I_{\mathcal{H}(S)} - AM^r_W)^{-1}B(I_n)$. Then

$$\begin{aligned}CY &= CM^r_W(I_{\mathcal{H}(S)} - AM^r_W)^{-1}B(I_n)\\ &= C(I_{\mathcal{H}(S)} - M^r_W A)^{-1}M^r_W B(I_n)\\ &= (B(I_n)W)^{\triangle}(W)\\ &= \mathbf{p}_0\ \left((S - S_{\{0\}})Z^*W(I_n - Z^*W)^{-1}\right)\\ &= \mathbf{p}_0\ \left(S(Z^*W - I_n + I_n)(I_n - Z^*W)^{-1}\right)\\ &= \left(\mathbf{p}_0\ S(I_n - Z^*W)^{-1}\right) - (\mathbf{p}_0\ S)\\ &= S^{\triangle}(W) - S_{\{0\}},\end{aligned}$$

from which the realization formula (5.16) follows. □

Remark 5.3 We are lacking a uniqueness result for (5.16).

Remark 5.4 The preceding analysis is still valid in the case of block $n \times n$ matrices whose entries are themselves matrices of size $p \times p$ or even operators. We did not consider this case to lighten the notation.

Remark 5.5 The "point evaluations" $F^{\wedge}(W)$ and $F^{\triangle}(W)$ allow to consider in the setting of (block) upper triangular matrices all interpolation problems considered for functions analytic in the open unit disk.

References

[1] D. Alpay. *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, volume 6 of *Panoramas et Synthèses*. Société Mathématique de France, 1998.

[2] D. Alpay, V. Bolotnikov, A. Dijksma, and Y. Peretz. A coisometric realization for triangular integral operators. Submitted.

[3] D. Alpay, V. Bolotnikov, A. Dijksma, J. Rovnyak, and C. Sadosky. Espaces de Hilbert inclus contractivement dans l'espace de Hardy du bidisque. *C. R. Acad. Sci. Paris Sér. I Math.*, 326:1365–1370, 1998.

[4] D. Alpay and P. Dewilde. Time varying signal approximation and estimation. In M. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Proceedings of the international symposium MTNS 89*, volume 5 of *Progress in systems and control theory*, pages 1–22. Birkhäuser Verlag, Basel, 1990.

[5] D. Alpay, P. Dewilde, and H. Dym. *Lossless inverse scattering and reproducing kernels for upper triangular operators*, volume 47 of *Operator Theory: Advances and Applications*, pages 61–133. Birkhäuser Verlag, Basel, 1990.

[6] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, volume 96 of *Operator theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[7] D. Alpay and Y. Peretz. Realizations for Schur upper triangular operators. In A. Dijksma, I. Gohberg, M. Kaashoek, and R. Mennicken, editors, *Contributions to operator theory in spaces with an indefinite metric*, volume 106 of *Operator Theory: Advances and Applications*, pages 37–90. Birkhäuser Verlag, Basel, 1998.

[8] D. Alpay and V. Vinnikov. Analogues d'espaces de de Branges sur des surfaces de Riemann. *C. R. Acad. Sci. Paris Sér. I Math.*, 318:1077–1082, 1994.

[9] T. Andô. de Branges spaces and analytic operator functions. Lecture notes, Hokkaido University, Sapporo, 1990.

[10] L. de Branges and J. Rovnyak. Canonical models in quantum scattering theory. In C. Wilcox, editor, *Perturbation theory and its applications in quantum mechanics*, pages 295–392. Wiley, New York, 1966.

[11] L. de Branges and J. Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York, 1966.

[12] H. Dym. *J contractive matrix functions, reproducing kernel spaces and interpolation*, volume 71 of *CBMS Lecture Notes*. Amer. Math. Soc., Providence R.I., 1989.

[13] P.A. Fillmore and J.P. Williams. On operator ranges. *Adv. in Math.*, 7:254–281, 1971.

[14] W. Rudin. *Analyse réelle et complexe*. Masson, Paris, 1980.