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Examining Young's modulus for wood

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Abstract

Symmetry considerations, dimensional analysis and simple approximations are used to derive a formula for Young's modulus of a simple anisotropic system, a straight-layer wood bar whose fibre axis makes an angle with respect to the bar's longitudinal axis. Agreement between the derived formula and experiment (carried out in far from ideal conditions) is within 10%. Improvements and extensions are suggested for this undergraduate physics experiment.

1. Introduction

Anisotropic properties of crystals are often avoided or just mentioned in passing in undergraduate physics programmes. Anisotropic behaviour is not examined because of the cost and experimental difficulties in working with anisotropic crystals and also because of the complexity and conceptual difficulties (i.e. visualizing the various axes, surfaces, special directions and symmetries in describing anisotropic physics).

A very effective way to overcome these practical and pedagogic problems is to use straight-layer wood as an anisotropic, large 'crystal'. Wood is inexpensive and easy to work with. Since wood is symmetric about any plane containing the fibre axis and any plane perpendicular to the fibre axis, the simple axial symmetry about the wood fibre axis allows a clear visualization of the anisotropic physics. Anisotropic properties of wood can readily be used to demonstrate the underlying principles in electric polarization physics [1] (dielectric permittivity and electric resistivity), in optics [2] (birefringence) and in thermal expansion [3] (thermal resistivity).

A general technical framework for describing direction-dependent properties is tensor mathematics [4]. A physical tensor quantity is defined by how it transforms under a coordinate transformation. A zero-rank tensor is just a scalar—a quantity which remains constant under a coordinate transformation. A first-rank tensor transforms like a vector under a coordinate change. A second-rank tensor transforms like a multiple of two vectors. An nth rank tensor transforms like a multiple of n vectors.

Physical properties which are scalars in isotropic systems become (or become related to) higher-rank tensors in anisotropic systems. For example, dielectric permittivity and thermal

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resistivity which, in isotropic systems, are constants become, in anisotropic systems, second-rank, symmetric tensors with respect to coordinate rotations.

Because of the simple symmetry of wood, the tensor technicalities are minimal—identifying and relating the various tensor properties, then analysing how they transform under a rotation. Here we examine, theoretically and experimentally, an elastic property of wood—Young's modulus—with a simple anisotropic system: vibrating wood bars cut with different angles between the wood fibre direction and the longitudinal, bar axis. The relevant physics requires a generalization of Hooke's law which, for a longitudinally-symmetric, isotropic system, states that, below the elastic limit, the strain (change in length per unit length) varies linearly with the stress (deforming force per unit area). The ratio of the stress to the strain is the stiffness constant or Young's modulus or spring constant in the case of a coil. The inverse of the stiffness constant is called the compliance constant.

Now consider a solid body subjected to a homogeneous stress (with no body torques) characterized by the symmetric, second-rank stress tensor [5] σ_{ij}. Below the elastic limit the resulting homogeneous strain, characterized by the symmetric, second-rank strain tensor [6] ε_{ij}, is such that there is a generalized Hooke law [7]

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} c_{ijkl}\varepsilon_{kl} \qquad i, j, k, l = 1, 2, 3 \tag{1}$$

where the 81 stiffness constants c_{ijkl} are components of a fourth-rank tensor. An alternative formulation of the connection between the stress and the strain is

$$\varepsilon_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} s_{ijkl}\sigma_{kl} \qquad i, j, k, l = 1, 2, 3 \tag{2}$$

where s_{ijkl} are the elastic compliances.

Under an orthogonal transformation of the coordinate axes (where the 'old' orthonormal basis vectors $\mathbf{1}, \mathbf{2}, \mathbf{3}$ change into the 'new' basis vectors $\mathbf{1}', \mathbf{2}', \mathbf{3}'$) the fourth-rank compliance tensor with components s_{ijkl} transforms into a new compliance tensor with components s'_{ijkl} according to

$$s'_{ijkl} = \sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\sum_{p=1}^{3} a_{im}a_{jn}a_{ko}a_{lp}s_{mnop} \qquad i, j, k, l, m, n, o, p = 1, 2, 3 \tag{3}$$

where the $a_{ij} \equiv (\mathbf{i}' \cdot \mathbf{j})$ are the direction cosines of the new axes relative to the old axes. Equation (3) is what characterizes s_{ijkl} as a fourth-rank tensor under a coordinate rotation—it transforms like a multiple of four vectors. This looks to be rather complicated, but simplifies drastically for wood.

Consider a bar of straight-layer wood with a fibre direction (designated as $\mathbf{2}$) that makes an angle θ with respect to the bar's longitudinal direction (designated as $\mathbf{2}'$), as in figure 1. Suppose a stress is applied to the bar in the transverse, $\mathbf{3}'$ direction. Then the only non-zero component of the stress tensor σ'_{ij} is σ'_{33}. Hence,

$$\varepsilon'_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} s'_{ijkl}\sigma'_{kl} = s'_{ij33}\sigma'_{33} \qquad i, j, k, l = 1, 2, 3. \tag{4}$$

Young's modulus $Y(\theta)$ for the transverse direction $\mathbf{3}'$ of a bar of given fibre orientation (specified by the angle θ) is defined as the ratio of the transverse stress σ'_{33} to the transverse strain ε'_{33}. Thus, by equation (4):

$$Y(\theta) \equiv \sigma'_{33}/\varepsilon'_{33} = 1/s'_{3333}. \tag{5}$$

Combining equations (3) and (5),

$$1/Y(\theta) = s'_{3333} = \sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{o=1}^{3}\sum_{p=1}^{3} a_{3m}a_{3n}a_{3o}a_{3p}s_{mnop} \qquad m, n, o, p = 1, 2, 3. \tag{6a}$$

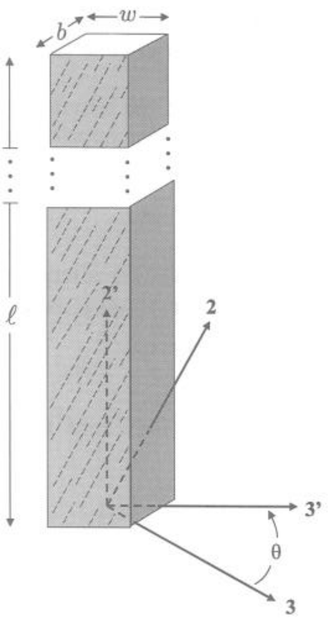

Figure 1. Wood bar of length ℓ, width w and thickness b with fibre direction **2** that makes an angle θ with respect to the longitudinal direction **2**′.

The only non-zero direction cosines are $\cos(\theta)$ and $\cos(90° - \theta)$ (which equals $\sin(\theta)$) and $\cos(180° - \theta)$ (which equals $-\cos(\theta)$). The general form of the possible fourth-order polynomials in the direction cosines of equation (6*a*) is then

$$1/Y(\theta) = a\cos^4(\theta) + p\sin^4(\theta) + q\cos^3(\theta)\sin(\theta) + r\cos(\theta)\sin^3(\theta) + c\sin^2(\theta)\cos^2(\theta) \quad (6b)$$

where a, p, q, r, c are constants.

Ideally (i.e. with no inhomogeneities), wood has infinite-fold symmetry about an axis, the fibre direction. Young's modulus $Y(\theta)$ must be symmetric in θ. Hence terms in equation (6*b*) that are not symmetric in θ must vanish. Setting q and r to zero results in

$$1/Y(\theta) = a\cos^4(\theta) + p\sin^4(\theta) + c\sin^2(\theta)\cos^2(\theta). \quad (7)$$

To proceed, one needs Young's modulus in terms of readily measured physical quantities. We utilize the connection between $Y(\theta)$ and the oscillation frequency of the rod for small angles when one of its ends is clamped and the other end is given a transverse tap (in the **3**′ direction). Instead of a detailed dynamical derivation [8], we present a dimensional analysis which yields the essential result.

Suppose the wood rod is viewed as a bundled-together collection of aligned elastic fibres with the oscillating bar length ℓ much greater than either transverse dimension, the rod width w or the thickness b. The frequency $\nu(\theta)$ for a bar of given fibre direction oscillating in the **2**′–**3**′ plane is expected to depend on Young's modulus $Y(\theta)$, the (presumed uniform) wood density ρ and the bar length ℓ. The simplest combination of these quantities that is dimensionally compatible is

$$\nu(\theta) \propto [Y(\theta)/\rho]^{1/2}/\ell. \quad (8)$$

To refine this a bit let us introduce an additional dependence on the rod width[1] w in the vibration plane (**2**′–**3**′):

$$\nu(\theta) \propto [Y(\theta)/\rho]^{1/2} F(w/\ell)/\ell \quad (9)$$

[1] Actually, the appropriate quantity is the radius of gyration in the (**2**′–**3**′) plane but this makes no difference for our purposes.

where F is a function of the small ratio (w/ℓ). The frequency is not expected to depend on the bar thickness b since the deformation of the wood is not perpendicular to the $(\mathbf{2}'–\mathbf{3}')$ vibration plane.

As the width w becomes negligibly small compared to the length ℓ, the frequency ν approaches zero. Hence $F(0)$ vanishes. For small (w/ℓ) we may approximate F by the first non-vanishing term of the Taylor expansion in (w/ℓ) about zero. Thus,

$$\nu(\theta) = d[Y(\theta)/\rho]^{1/2}[w/\ell^2] \tag{10}$$

where d is a numerical constant.

Combining equations (7) and (10), we find that the period of oscillation $\tau(\theta) \equiv (1/\nu(\theta))$ for a bar with fibres at an angle θ with respect to the bar's longitudinal axis is given by

$$\tau^2(\theta) = A\cos^4(\theta) + B\sin^4(\theta) + C\sin^2(\theta)\cos^2(\theta) \tag{11}$$

where

$$A \equiv (a\rho\ell^4)/(d^2w^2); \qquad B \equiv (p\rho\ell^4)/(d^2w^2); \qquad C \equiv (c\rho\ell^4)/(d^2w^2). \tag{12}$$

Since

$$\tau^2(\theta)/\tau^2(0°) = 1/[Y(\theta)/Y(0°)] \tag{13}$$

the behaviour of $Y(\theta)/Y(0°)$ follows from equation (11).

To fix the parameters A, B, C of equation (11), one needs the value of the period of oscillation for at least three angles. For example:

$$A = \tau^2(0°); \qquad B = \tau^2(90°); \qquad C = 4\tau^2(45°) - \tau^2(0°) - \tau^2(90°). \tag{14}$$

Alternatively, one may employ a 'best-fit' method to fix A, B and C.

2. Experimental details/results

Our wooden rods were cut roughly from a rectangular pine plate of dimensions 120 cm by 85 cm by 1.5 cm, purchased at a local lumber yard. The plate was itself a composite of bonded-together pine boards each about 8.5 cm between seams. Five bars, each 118 cm by 1.5 cm by 1.5 cm, were sawed with fibre angles 0°, 30°, 45°, 60°, 90° with respect to the longitudinal axis. As much as possible, knots were avoided. Any seam was kept at least 7 cm from the end of the bar that was to be fixed since the deformation would be greatest near the clamped region. For non-zero fibre angles, the 118 cm dimension was achieved by gluing on an additional bar section opposite to the end of the bar that was to be clamped. The higher-angle bars had a tendency to curve due to the unbalancing of internal forces in cutting the bars out of the plate.

The upper end of each vertical bar was tightly clamped over a 2 cm length using a heavy vice. The lower, free end of each bar was given a slight tap and the ensuing vibratory motion of the end of each bar was tracked by a motion detector coupled to a personal computer. We used a V-Scope Litek Model RCV-105 motion detector set in the vibration plane about 35 cm from the centre of vibration of the bar's free end. The maximum amplitude of vibration was kept below 2 cm— $<2\%$ of a bar's length. Thus the angle between the oscillating bar and the vertical direction remained very small.

A typical experiment output, for the fibre direction $\theta = 30°$, is shown in figure 2. Table 1 summarizes the measured periods $\tau(\theta)_{\text{measured}}$ for the various fibre angles.

An alternative method to measure the period is by a stroboscope.

3. Analysis/comments

To examine whether the measurements and the formula of equation (11) accommodate each other, we first minimized the function

$$\phi(A, B, C) \equiv \sum_{\text{measured angles}} [\tau^2(\theta)_{\text{measured}} - \tau^2(\theta)_{\text{theory}}]^2 \tag{15}$$

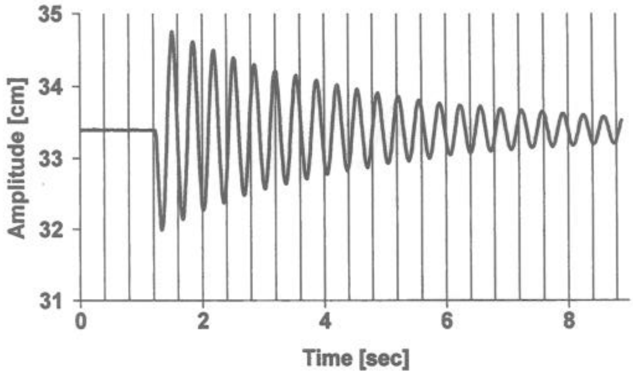

Figure 2. Displacement versus time of the lower end of a vibrating bar with its upper end clamped. The fibre direction is 30° with respect to the bar's longitudinal axis.

Table 1. The measured period of oscillation for various fibre directions.

θ (deg)	$\tau(\theta)_{\text{measured}}$ (s)
0, 180	0.114
30, 150, 210, 330	0.34
45, 135, 225, 315	0.53
60, 120, 240, 300	0.55
90, 270	0.72

with respect to the parameters A, B, C. This linear least square method yields

$$A = 0.018; \qquad B = 0.498; \qquad C = 0.421. \tag{16}$$

Using these values, the polar-coordinate plot of figure 3 shows $\tau(\theta)$ of equation (11) (as the radial component) for every angle θ between 0° and 360°. A $[1/Y(\theta)]^{1/2}$ versus θ plot will have the same shape as the $\tau(\theta)$ versus θ plot of figure 3. The measured periods are also shown in figure 3. Agreement between the 'best-fit' curve and the measurements is within 10%.

Needless to say, this experiment can be refined and improved (i.e. avoiding 'gluing together' sections of the bars, using many more bars with different fibre angles, etc), but our purpose here was to show that, even with conditions far from ideal, reasonable results can be achieved in examining Young's modulus for this anisotropic system.

It is interesting to consider how the overall shape of the $\tau(\theta)$ versus θ curve depends on the relative size of the A, B, C parameters. The two general shapes are a number 8-like shape—as in figure 3—and a clover-like shape where the top and bottom of an '8' are pushed toward the centre. The distinction between these shapes stems from the number of extrema of $\tau(\theta)$ (or $1/Y(\theta)$) as a function of θ. A straightforward calculation shows that $\tau(\theta)$ has extrema when

$$\sin(2\theta) = 0 \qquad \text{or} \qquad \tan^2\theta = (C - 2A)/(C - 2B). \tag{17}$$

The second condition is satisfied only if both the numerator and denominator have the same sign. For wood, the elastic forces along the fibre direction are much stronger than those perpendicular to the fibres. It follows that $A \ll B$ as is borne out by our data. Hence the second condition of equation (17) reduces to whether C is greater than $2B$. This was not the case for our bars but there may well be varieties of wood for which the second condition of equation (17) is satisfied, thus yielding a clover-shaped polar-coordinate $\tau(\theta)$ versus θ plot.

It is evident that in developing equation (10) we have tacitly assumed that the gravitational force acting on a bar is negligible compared to the elastic forces. Is this reasonable? If gravity were completely insignificant for the oscillating bar, then the following experiments should

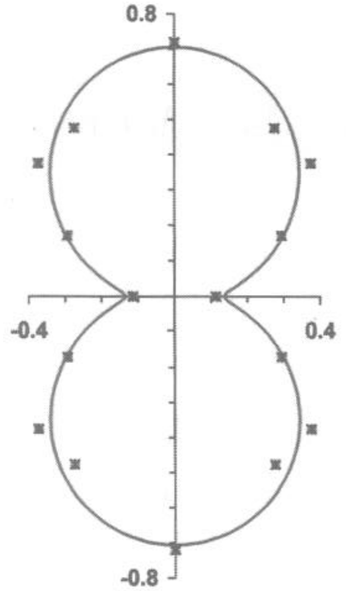

Figure 3. Polar-coordinate plot of the oscillation period $\tau(\theta)$ of equation (11) with the parameters A, B, C set to the values given in equation (16). The radial coordinate $\tau(\theta)$ is shown for fibre angles θ between 0° and 360°. Data points are shown as squares.

(This figure is in colour only in the electronic version)

yield identical results for the period of vibration of a bar of given fibre direction with respect to the bar's longitudinal axis.

(I) The vertical bar is clamped at the top end and the lower end is free. (In this case gravity works to help the elastic forces in returning the vibrating bar toward equilibrium.)

(II) The vertical bar is clamped at the bottom end and the top end is free. (In this case gravity works to oppose the elastic forces acting to return the vibrating rod to equilibrium.)

(III) The bar is kept horizontal with one end clamped. (In this case the bar remains in a gravitational equipotential and so gravity does not affect the rod's vibration.)

It turns out that for small fibre angles near 0° (and 180°) there is little variation in the measured period among the three methods, but as the fibre angle grows the respective vibration periods $\tau(\theta)_{\rm I}$, $\tau(\theta)_{\rm II}$, $\tau(\theta)_{\rm III}$ diverge somewhat in the expected order

$$\tau(\theta)_{\rm I} < \tau(\theta)_{\rm III} < \tau(\theta)_{\rm II}. \tag{18}$$

As the fibre angle approaches 90°, method III becomes nonviable due to the bending and fragility of a long bar. Using shorter bars would reduce this problem, but require measuring smaller oscillation periods.

It would be interesting to explore more precisely, both theoretically and experimentally, the elastic and gravitational effects that contribute to the period of oscillation. We leave this to the avid reader and his or her students.

Acknowledgments

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