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Production/Clearing Models Under Continuous and Sporadic Reviews

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Abstract. We consider production/clearing models where random demand for a product is generated by customers (e.g., retailers) who arrive according to a compound Poisson process. The product is produced uniformly and continuously and added to the buffer to meet future demands. Allowing to operate the system without a clearing policy may result in high inventory holding costs. Thus, in order to minimize the average cost for the system we introduce two different clearing policies (continuous and sporadic review) and consider two different issuing policies ("all-or-some" and "all-or-none") giving rise to four distinct production/clearing models. We use tools from level crossing theory and establish integral equations representing the stationary distribution of the buffer's content level. We solve the integral equations to obtain the stationary distributions and develop the average cost objective functions involving holding, shortage and clearing costs for each model. We then compute the optimal value of the decision variables that minimize the objective functions. We present numerical examples for each of the four models and compare the behaviour of different solutions.

Keywords: clearing policies, stationary distributions, optimization

AMS 2000 Subject Classification: 90B05 Inventory, storage, reservoirs; 90B22 Queues and service; 90B30 Production models

1. Introduction

We consider four versions of a production/clearing model where a single machine produces a certain product into a buffer continuously and uniformly. Customers (e.g., retailers) generate the demand for the product and they arrive according to a compound Poisson process with rate λ. The demand sizes are independent and identically distributed with distribution $G(\cdot)$ and mean $1/\mu$. In the absence of any controls (i.e.,

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clearing rules), the content level is generated by reflection on the deterministic production minus the demand process. Negative inventory is not allowed—that is, there is no backlogging—so that 0 is a reflecting barrier. Since production never stops, the content level fluctuates as a reflected continuous time random walk process whose sample path increases at rate 1 (without loss of generality) between negative jumps which are the demands sizes. In two of our models we assume that each demand can be either fully or partially satisfied ("all-or-some"). This may correspond a situation where it may be feasible for the customers to purchase the remaining units from a different supplier. In the other two models the demand processes are of the all-or-nothing type ("all-or-none"). That is, a demand is either satisfied or completely unsatisfied if its size is greater than the content level. This may correspond to a "mission-critical" situation where the customers must receive exactly what they need, or they may go elsewhere. In the language of stochastic insurance models such content level processes are generally called *risk processes*.

Obviously, in the absence of any controls (clearing rules), the content level process of the buffer is a regenerative process if and only if the constant production rate is less than the demand rate, i.e., $1 < \lambda/\mu$. Furthermore, in the absence of any controls the content level process can be interpreted as the conditional elapsed waiting time (EWT) process of the basic $G/M/1$ queue given that the idle periods are deleted; for more details see Perry and Posner (1990, 2002).

The four models studied in this paper are characterized by four different types of controls according to the following possibilities assuming that the controller (e.g., inventory manager) has the option to sell the product to a large wholesaler:

1. Each time a pre-determined control limit level q is reached by the content level, the buffer is cleared and transferred to the wholesaler. In the language of inventory theory, the buffer is controlled under *continuous review* because the buffer is observed continuously over time and the controller "sees" the content once level q is reached. For this policy, q is the controller's decision variable.
2. The wholesaler is not obligated to a fixed time schedule or to a fixed amount to clear the buffer. He does this whenever he happens to be, by chance, on the spot. Under this *sporadic review* control policy the buffer is cleared according to a random arrival process of the controller. We assume it is a Poisson process with rate ξ. For this policy, ξ is the controller's decision variable.

Under the *continuous review* control policy the wholesaler must be continuously available. This results in high running and transfer costs. The *sporadic review* control policy enables the wholesaler to be "unreliable" in the sense that he clears the system sporadically in accordance with his convenience.

We consider four different models. The first two models (which will be denoted as Model I and Model II) correspond to the *continuous review*. The other two models (which will be denoted by Model III and Model IV) correspond to the *sporadic review*. In Models I and III we assume that the demand is completely or partially satisfied, i.e., the issuing policy is such that "All-or-Some" of the demand is met. In Models II and IV we assume that the "All-or-None" issuing policy results in either all or none of the demand being satisfied. That is, each demand can be either completely satisfied or completely unsatisfied depending on the content level present. See Table 1 for a summary of the characteristics of the four models.

For the four models we assume the following types of costs: (i) Proportional holding cost h per unit item and unit time [$/unit-time]; (ii) a proportional shortage cost π [$/unit] for each unsatisfied demand, and (iii) a fixed cost K [$] of clearing the buffer.

For background on clearing models see Stidham (1974, 1977, 1986), and Serfozo and Stidham (1978). Further uses of these models include the control of epidemics, in which the quantity of interest is the number of susceptibles and the clearing corresponds to mass vaccination; see, Perry and Stadje (2001). Another point of view of clearing models has recently been developed in the queueing literature where, in addition to regular customers, so-called "negative arrivals" are also considered. A negative arrival has the effect of deleting some amount of the workload from the queue. Such models were first studied by Boucherie and Boxma (1996), Gelenbe and Glynn (1991) and Harrison and Pitel (1993, 1996).

REMARK 1 It was indicated above that the demand sizes have a general distribution $G(\cdot)$. The analytic results in that case can be found in a similar manner to that of Perry et al. (2001). However, the expressions of the relevant functionals, which become the components of the objective function in the optimization models, are completely intractable for optimization purposes. In this study we therefore restrict our attention to the case of $G(x) = 1 - e^{-\mu x}$ for $x > 0$. As will be seen in subsequent analysis, the solutions of the steady state densities as well as the Laplace transform (LT) of the clearing cycle can be quite complicated. Even the simple extensions beyond the exponential case, namely, the expressions of the simple Coxian or hyperexponential distributions become too cumbersome for optimization purposes. Nevertheless, there is one case in which the general jump size is tractable, as will be seen in Section 4 below.

The remainder of the paper is structured as follows: In Sections 2, 3, 4 and 5 we introduce the four models I, II, III and IV, respectively. In these sections, in addition to the probabilistic analysis of the content level process that requires the (analytic or numerical) solution of an integral equation, we also present optimization models to minimize a suitable average cost function. The paper ends in Section 6 with a summary and comparison of the four models.

Table 1. Four model types arising from different combinations of timing of the reviews (clearing policies) and the way the demand is met (issuing policies).

		Issuing Policy	
		"All-or-Some"	"All-or-None"
Clearing Policy	Continuous Review	Model I	Model II
	Sporadic Review	Model III	Model IV

2. Model I: Continuous Review with Completely or Partially Satisfied ("All-or-Some") Demand

We start by constructing the dynamics of the buffer process of Model I. Let $\mathbf{N} = \{N(t): t \geq 0\}$ be a Poisson process with rate λ and $S_1, S_2, \ldots$ be i.i.d. random variables with mean $1/\mu$ which are also independent of $\mathbf{N}$. The so-called *risk version* of the continuous time random walk is the process $\mathbf{X} = \{X(t) : t \geq 0\}$ where

$$X(t) = t - \left(S_1 + S_2 + \cdots + S_{N(t)}\right). \tag{1}$$

As we indicated before, we restrict our attention to the case where $S_i \sim \exp(\mu)$ (but, see Remark 3 below). Next, consider the reflected process $\mathbf{W} = \{W(t) : t \geq 0\}$ where

$$W(t) = X(t) - \min_{0 \leq s \leq t} X(s) \tag{2}$$

and define the stopping time $\tau_q = \inf\{t : W(t) \geq q\}$ for some constant q. The content level process of the buffer $\mathbf{V}_1 = \{V_1(t) : t \geq 0\}$ is a regenerative process with cycle time τ_q whose sample path is the stochastic replication of the family $\{W(t) : 0 \leq t \leq \tau_q\}$. Typical realizations of $\mathbf{V}_1$ and τ_q are depicted in Figure 1.

2.1. Stationary Distribution of the Content Level $V_1(t)$

We are interested in the stationary distribution of the content level process and cycle length τ_q which is the time between clearings.

Being a regenerative process with finite expected cycle length τ_q, the $\mathbf{V}_1$ process possesses a steady state density $f_1(x) \equiv f_{V_1}(x)$ whose Khintchine–Pollaczeck integral equation is given by

$$f_1(x) = \lambda \int_x^q e^{-\mu(w-x)} f_1(w)dw + f_1(q), \quad 0 \leq x \leq q. \tag{3}$$

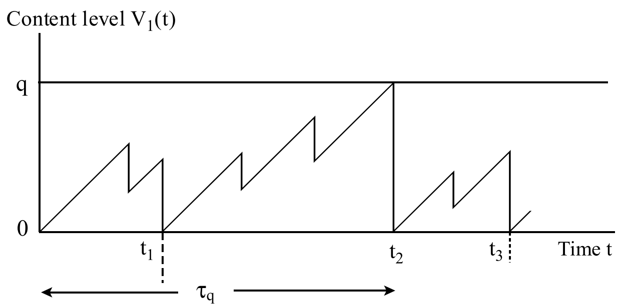

Figure 1. Sample paths for the inventory level in Model I [continuous review with completely or partially satisfied ("All-or-Some") demand]. At epoch t_2 the buffer is cleared and at epochs t_1 and t_3 some of the demand at the buffer (indicated by the dotted lines) is lost. The cycle length is τ_q.

The fact that the steady state distribution of V_1 is absolutely continuous follows from Level Crossing Theory (LCT); see, Cohen (1977) and Doshi (1992). Also, it follows from LCT that the steady state density is the long-run average number of down-crossings of level $x > 0$. In the sequel we assume that the limiting density has a derivative. This assumption is verified by the numerical analysis since by the limiting theorem for regenerative processes the steady state density $f_1(x)$ is unique. The steady state density can also be interpreted as the long-run average number of up-crossings of level x. Thus, the right hand side of (3) must be equal to the long-run average number of down-crossings of the same level x (since, by assumption, V_1 is an ergodic process). As a result, the downward jumps consist of two types, Poisson jumps and clearing jumps. The Poisson jumps occur whenever $\mathbf{V}_1 \in (x, q)$ just before arrival and the conditional probability to jump below x given $\mathbf{V}_1 \in dw$ is $e^{-\mu(w - x)}$. Furthermore, the limiting density of $\mathbf{V}_1$ just before a Poisson downward jump and the steady state density $f_1(\cdot)$ coincide by PASTA (see; Perry and Posner (1989, 2002) and Wolff (1989). The second term on the right hand side of (3) is $f_1(q)$. Again, by LCT, $f_1(q)$ is the long-run average number of clearings (by assumption the buffer is clear each time $\mathbf{V}_1$ reaches level q). But each clearing is automatically a down-crossing for all $x \in (0, q)$.

In order to solve the integral equation (3) for $f_1(x)$, we differentiate (3) twice w.r.t. x and obtain

$$f_1''(x) + (\lambda - \mu) f_1'(x) = 0. \tag{3}$$

Solving this second order linear ordinary differential equation (ODE), we have

$$f_1(x) = ae^{-(\lambda-\mu)x} + b, \quad 0 \le x \le q \tag{4}$$

for some constants a and b. To determine a and b note that by (4) we have $f_1(0) = a + b$ and $f_1(q) = ae^{-(\lambda-\mu)q} + b$. Together with the normalizing condition $\int_0^q f_1(x)\,dx = 1$ this implies

$$a = \frac{\lambda(\lambda - \mu)}{\lambda[1 - e^{-(\lambda-\mu)q}] - \mu q(\lambda - \mu)e^{-(\lambda-\mu)q}}, \tag{5}$$

and

$$b = -\frac{1}{\lambda}\left[a\mu e^{-(\lambda-\mu)q}\right]. \tag{6}$$

Remark 2 When $\mu = \lambda$, the second order ODE reduces to $f_1''(x) = 0$ which gives $f_1(x) = \hat{a}x + \hat{b}$ as the steady-state density. To determine the constants $\hat{a}$ and $\hat{b}$, we use the integral equation (3) and the normalizing condition $\int_0^q f_1(x)\,dx = 1$ and obtain

$$\hat{a} = -\frac{2}{q}\left(\frac{\mu}{2 + \mu q}\right), \quad \hat{b} = \frac{2}{q}\left(\frac{1 + \mu q}{2 + \mu q}\right).$$

At the endpoints, the stationary density assumes the values

$$f_1(0) = \frac{2}{q}\left(\frac{1+\mu q}{2+\mu q}\right) \quad \text{and} \quad f_1(q) = \frac{2}{q}\left(\frac{1}{2+\mu q}\right).$$

Finally, by LCT, the clearing rate is $f_1(q)$ so that the expected time between clearings, i.e., the expected cycle time is

$$E(\tau_q) = \frac{1}{f_1(q)}. \tag{7}$$

Similarly, the rate of the unsatisfied demand is found as

$$f_1(0) - f_1(q) = \frac{2\mu}{2+q\mu}$$

so that the expected time between two unsatisfied demands is $(2 + q\mu) / (2\mu)$.

To compute the Laplace transform of τ_q we define for simplicity the process $\mathbf{Y} = \{Y(t) : t \geq 0\}$ where $Y(t) = q - X(t)$. Then, we define the stopping time $T_q = \inf\{t : Y(t) = 0 \text{ or } Y(t) \geq q\}$.

To relate the Laplace transform of T_q to that of τ_q define

$$\phi_*(\beta) = E\left(e^{-\beta T_q} \cdot \mathbf{1}_{\{Y(T_q)=0\}}\right) \quad \text{and} \quad \phi^*(\beta) = E\left(e^{-\beta T_q} \cdot \mathbf{1}_{\{Y(T_q)\geq q\}}\right)$$

and let $\Gamma_q(\beta) = E\left(e^{-\beta\tau_q}\right)$ be the LT of τ_q. Then,

$$\Gamma_q(\beta) = \frac{\phi_*(\beta)}{1-\phi^*(\beta)}.$$

To see this note that if the event $\{Y(T_q) = 0\}$ occurs, level 0 is reached by $\mathbf{Y}$ before level q is up-crossed; thus, $T_q = \tau_q$. If the event $\{Y(T_q) \geq q\}$ occurs, level q is up-crossed before level 0 is reached by $\mathbf{Y}$ and the process $\mathbf{V}_1$ regenerates itself. In terms of LTs we obtain the renewal equation

$$\Gamma_q(\beta) = \phi_*(\beta) + \phi^*(\beta)\Gamma_q(\beta).$$

Solving for $\Gamma_q(\beta)$ the result follows.

Thus, in order to find the LT of τ_q we only have to compute $\phi_*(\beta)$ and $\phi^*(\beta)$. To this end, we use the Wald's martingale

$$M(t) = \frac{e^{-\alpha Y(t)}}{E[e^{-\alpha Y(t)}]} - e^{-\alpha Y(0)}$$

as employed in Perry and Posner (1989) and apply the optional sampling theorem to the stopping time T_q (obviously, the conditions for using the optional sampling theorem hold because $\{Y(t) : 0 \leq t < T_q\} \in [0,q]$). Thus, $E[M(0)] = E[M(T_q)]$ yields

$$e^{-\theta_i(\beta)q} = E\left[e^{-\theta_i(\beta)Y\left(T_q\right)-\beta T_q}\right], \quad \text{for } i = 1, 2. \tag{8}$$

Obviously, we have

$$\theta_1(\beta) = \frac{1}{2}\left[-(\mu - \lambda - \beta) + \sqrt{(\mu - \lambda - \beta)^2 + 4\beta\mu}\right],$$

and

$$\theta_2(\beta) = \frac{1}{2}\left[-(\mu - \lambda - \beta) - \sqrt{(\mu - \lambda - \beta)^2 + 4\beta\mu}\right].$$

By using the above results, we have

$$\begin{aligned} e^{-\theta_i(\beta)q} &= E\left(e^{-\theta_i(\beta)Y\left(T_q\right)-\beta T_q}\right) \\ &= \phi_*(\beta) + E\left(e^{-\theta_i(\beta)Y\left(T_q\right)-\beta T_q}\cdot\mathbf{1}_{\left\{Y\left(T_q\right)q\right\}}\right), \quad i = 1, 2. \end{aligned} \tag{9}$$

The fact that the jumps of $\mathbf{Y}$ are exponentially distributed implies that given the event $\{Y(T_q) \geq q\}$, the overflow above q is also exponentially distributed by the memoryless property of the exponential distribution, and that $Y(T_q)$ and T_q are conditionally independent. Thus,

$$e^{-\theta_i(\beta)q} = \phi_*(\beta) + \frac{\mu}{\mu + \theta_i(\beta)} e^{-\theta_i(\beta)q}\phi^*(\beta), \quad i = 1, 2. \tag{10}$$

The fundamental identity (10) represents two equations with two unknowns; $\phi_*(\beta)$ and $\phi^*(\beta)$. Solving for $\phi_*(\beta)$ and $\phi^*(\beta)$ we obtain

$$\phi_*(\beta) = \frac{e^{-[\theta_1(\beta)+\theta_2(\beta)]q}\left(\dfrac{\mu}{\mu + \theta_2(\beta)} - \dfrac{\mu}{\mu + \theta_1(\beta)}\right)}{\dfrac{\mu e^{-\theta_2(\beta)q}}{\mu + \theta_2(\beta)} - \dfrac{\mu e^{-\theta_1(\beta)q}}{\mu + \theta_1(\beta)}}$$

and

$$\phi^*(\beta) = \frac{e^{-\theta_2(\beta)q} - e^{-\theta_1(\beta)q}}{\dfrac{\mu e^{-\theta_2(\beta)q}}{\mu + \theta_2(\beta)} - \dfrac{\mu e^{-\theta_1(\beta)q}}{\mu + \theta_1(\beta)}}.$$

Since the Laplace transform $\Gamma_q(\beta)$ of τ_q is obtained in terms of the LTs $\phi_*(\beta)$ and $\phi^*(\beta)$ for which we have explicit expressions, the moments of τ_q can be easily calculated by using $\Gamma_q(\beta)$. One can also determine the distribution of τ_q using numerical techniques as suggested by Abate and Whitt (1995).

REMARK 3 The assumption $S_i \sim \exp(\mu)$ can be extended technically to some special cases of phase-type distributions; for examples of hyperexponential and Coxian distributions, see Boxma et al. (2001). More general cases have an importance only from a theoretical point of view; see, Perry et al. (2001) for the analysis of the general case. However, the expressions of the functionals obtained in the general case are too cumbersome and intractable for optimization or sensitivity analysis.

2.2. Optimization of Model I

Having obtained the stationary distribution $f_1(x) \equiv f_{V_1}(x)$ of the content level process $\mathbf{V}_1(t)$, we now present an optimization model to determine the optimal value of the clearing level q that minimizes the average cost incurred by the system. In order to form the objective function, we make use of the renewal-reward theorem (Ross, 1983) and compute the average cost (i.e., the cost rate) $C_1(q)$ for Model I as

$$\text{Average Cost}: \quad C_1(q) = \frac{E(\text{cycle cost})}{E(\text{cycle length})}.$$

For Model I, the expected cycle length is simply $1/f_1(q)$ as indicated by (7). We now develop expressions for the average cost which includes (i) holding cost, (ii) shortage cost for unsatisfied demand, and (iii) fixed cost of clearing the buffer.

The expected holding cost in a cycle is obtained as

$$E(\text{holding cost per cycle}) = E(HC) = h\left[\int_0^q x\, f_1(x)\, dx\right]\left(\frac{1}{f_1(q)}\right),$$

which is the product of holding cost per unit per time [$/unit-time] and expected inventory level [unit] and expected cycle length [time]. Dimensional analysis reveals that the dimension of $E(HC)$ is [$], as required.

To calculate the expected shortage cost first note that the effective rate of arrivals whose demands are not satisfied is $\lambda\int_0^q [1 - G(x)]f_1(x)\, dx$ with dimension [arrival/time]. In this "All-or-Some" case, when demand exceeds the available number units in the buffer, only the difference between the demand and available supply is lost (that is, unsatisfied). Thus, the expected value of the unsatisfied demand *given* that demand exceeds supply equals the "overshoot" $1/\mu$ from the memoryless property of the exponential. Hence, the expected shortage cost is

$$E(\text{shortage cost}) = E(SC) = \pi\left\{\lambda\int_0^q [\mathbf{1} - G(x)]f_1(x)\, dx\right\}\left(\frac{1}{\mu}\right)\frac{1}{f_1(q)},$$

with dimension [$].

The expected cost of clearing the buffer in a cycle is simply

$$E(\text{clearing cost}) = E(CC) = K$$

with dimension [$]. Combining the above results, we find the average cost per time (i.e., the cost rate) for Model I as

$$C_1(q) = h\int_0^q x f_1(x)\, dx + \frac{\pi\lambda}{\mu}\int_0^q [1 - G(x)] f_1(x)\, dx + K f_1(q)$$

with dimension [$/time], as required.

As a simple example consider the case with parameters $(\lambda, \mu; h, \pi, K) = (5,10; 1, 2, 4)$ which gives rise to the strictly convex cost function $C_1(q)$ in Figure 2. Differentiating $C_1(q)$ and solving $C_1'(q) = 0$ we find the optimal solution as $q^* = 2.15$ which results in a minimum average cost of $C_1(q^*) = 2.05$. For $q^* = 2.15$, the stationary density of the buffer level is found as $f_1(x) = -0.5 \times 10^{-5} e^{5x} + 0.48$ which is plotted in Figure 3 for $x \in (0, q^*)$.

The intuition behind the shape of $f_1(x)$ as displayed in Figure 3 is the following: When the process is close to $q^* = 2.15$, it will quickly jump away to zero with high probability so that the density is small in the neighborhood of 2.15. Moreover, the density is decreasing since the sojourn time is smaller for an interval of states that are located farther away from zero.

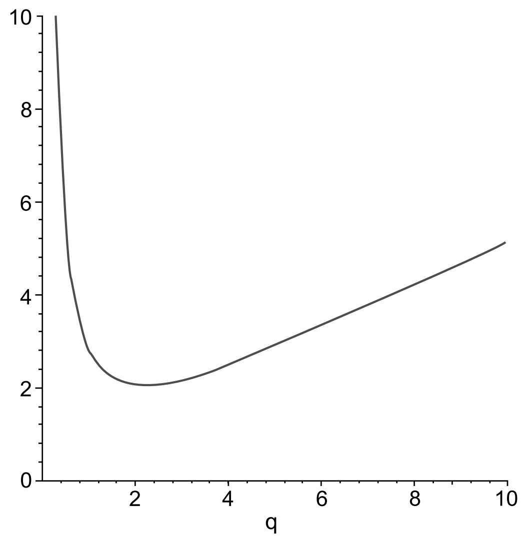

Figure 2. Average cost function $C_1(q)$ for Model I when $(\lambda,\ \mu; h, \pi, K) = (5,10; 1, 2, 4)$. This function is minimized at $q^* = 2.15$ which gives $C_1(q^*) = 2.05$.

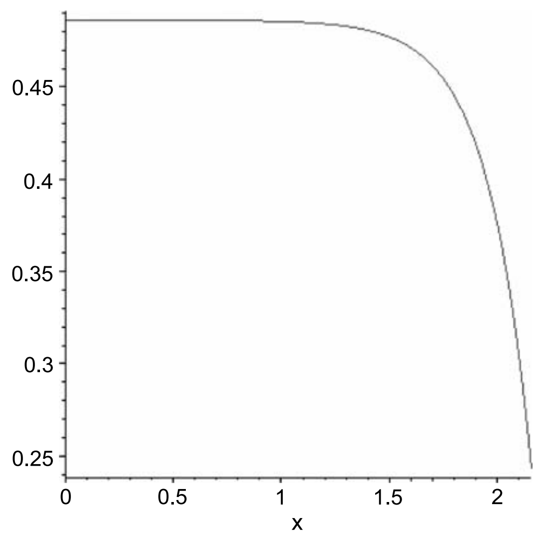

Figure 3. Stationary density $f_1(x)$ of the content level process when $q^* = 2.15$.

3. Model II: Continuous Review with Completely Satisfied or Completely Unsatisfied ("All-or-Nothing") Demand

In Model II the content level process $\mathbf{V}_2 = \{V_2(t) : t \geq 0\}$ of the buffer is no longer a continuous time random walk. The demands are of the "All-or-Nothing" type in the sense that each demand is either completely satisfied (if its size is smaller than the content level); or otherwise, completely unsatisfied. For a sample path of the buffer process, see Figure 4.

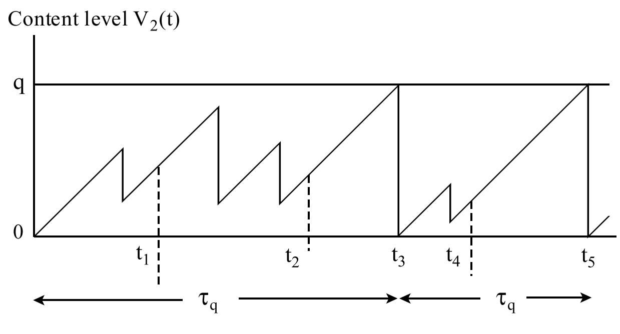

Figure 4. Sample paths for the inventory level in Model II [continuous review with completely satisfied or completely unsatisfied ("All-or-Nothing") demand]. At epochs t_3 and t_5 the buffer is cleared and at epochs t_1, t_2 and t_4 all demand at the buffer (indicated by the dotted lines) is lost. The cycle length is τ_q.

3.1. Stationary Distribution of the Content Level $V_2(t)$

The $\mathbf{V}_2$ process possesses a steady state density $f_2(x) = f_{V_2}(x)$ whose steady state (Khintchine–Pollaczeck) integral equation is given by

$$f_2(x) = \lambda \int_x^q \left[e^{-\mu(w-x)} - e^{-\mu w} \right] f_2(w)\, dw + f_2(q), \quad 0 \le x \le q. \tag{11}$$

Again, the left hand side of (11) is the long-run average number of up-crossings of level x. Thus, the right hand side must be the long-run average number of down-crossings. The parameter λ is the rate of the Poisson downward jumps and

$$\Pr(w - x \le S < w) = e^{-\mu(w-x)} - e^{-\mu w}$$

is the probability that the demand is completely satisfied (because it is greater than x but less than the content level).

To solve for $f_2(x)$ we take the derivative in both sides of (11) and get

$$f_2'(x) = \lambda \mu e^{\mu x} \int_x^q e^{-\mu w} f_2(w)\, dw - \lambda(e^{\mu x} - 1) e^{-\mu x} f_2(x),$$

or,

$$e^{-\mu x} f_2'(x) = \lambda \mu \int_x^q e^{-\mu w} f_2(w)\, dw - \lambda \left(e^{-\mu x} - e^{-2\mu x} \right) f_2(x)$$

where $f_2'(\cdot)$ $[f_2''(\cdot)]$ denotes the first [second] derivative of $f_2(\cdot)$ w.r.t. x. Taking the second derivative we obtain the following second order ODE

$$f''_2(x) + [\lambda(1 - e^{-\mu x}) - \mu] f_2'(x) + 2\lambda \mu e^{-\mu x} f_2(x) = 0. \tag{12}$$

This ODE has two boundary conditions: (i) $f_2(0) = f_2(\mathrm{q})$ since in the long-run the number of hittings of level q is equal to that of level 0+, and, (ii) the normalizing condition for the distribution, i.e., $\int_0^q f_2(x) dx = 1$. Using these conditions equation (12) can be, in principle, solved to yield the stationary distribution of the buffer level which can be used in the optimization of the average cost function. To solve the second order ODE with variable coefficients (12), we have used the computer algebra system Maple (Char, 2002) and obtained the result in terms of two arbitrary constants and a complicated expression involving an integral which cannot be evaluated explicitly. Since the first boundary condition $f_2(0) = f_2(q)$ itself is not predetermined and its value must be chosen so that $\int_0^q f_2(x)\, dx = 1$, we decided to implement a numerical scheme to solve this ODE directly as will be discussed below.

3.2. *Optimization of Model II*

For Model II, we again form the average cost objective function as the ratio of expected cycle cost to the expected cycle length which is $1/f_2(0) = 1/f_2(q)$. As in Model I, the expected holding cost per cycle is

$$E(\text{holding cost per cycle}) = E(HC) = \frac{h}{f_2(0)} \int_0^q x\, f_2(x)\, dx.$$

To calculate the expected shortage cost note that, as before, the effective rate of arrivals whose demands are not satisfied is $\lambda \int_0^q [1 - G(x)] f_2(x)\, dx$ with dimension [arrival/time]. However, in this "All-or-Nothing" case, when demand exceeds the available content level (or alternatively, number of items) in the buffer, the total demand is completely lost (unsatisfied). Thus, the expected value of the unsatisfied demand given that demand exceeds supply equals the expected level of the buffer $I_2(q) \equiv \int_0^q x\, f_2(x)\, dx$ plus the negative "overshoot" that from the memoryless property of the exponential distribution is equal to $1/\mu$. Hence, the expected shortage cost is

$$E(\text{shortage cost}) = E(SC) = \frac{\pi\lambda}{f_2(0)} \left[\frac{1}{\mu} + I_2(q)\right] \int_0^q [\mathbf{1} - G(x)] f_2(x)\, dx.$$

As in Model I, the expected cost of clearing the buffer in a cycle is simply

$$E(\text{clearing cost}) = E(CC) = K.$$

Combining the above results, we find the average cost per time (i.e., the cost rate) for Model II as

$$C_2(q) = h \int_0^q x\, f_2(x)\, dx + \pi\lambda \left[\frac{1}{\mu} + I_2(q)\right] \int_0^q [1 - G(x)] f_1(x)\, dx + K\, f_2(0).$$

The optimization of this cost function is substantially more challenging than the one we encountered in Model I. Since the stationary density $f_2(x)$ must first be solved to calculate the average cost $C_2(q)$, we use the following procedure to find the optimal q: (i) Start with a value of q that is likely to be near optimal, (ii) Guess a value of a and solve the ODE (12) numerically to determine $f_2(x)$ for $0 \leq x \leq q$, (iii) Since a may not have been chosen correctly and thus the density may not integrate to 1, use the "shooting method" (Roberts and Shipman, 1972) iteratively to determine the correct value of a so that $\int_0^q f_2(x)\, dx = 1$, (iv) Evaluate the average cost at the q value that was chosen, (v) Use a line search method such as dichotomous search (Bazaraa and Shetty, 1979, Ch. 8) and examine different values of q (and find corresponding values of a) to find the optimal q.

Implementing this procedure we found the results as shown in Table 2 which gives the optimal value as $q^* = 2.20$ with $C_2(q^*) = 2.48$. (All results are to two significant digits.) The density $f_2(x)$ corresponding to $q^* = 2.20$ is shown in Figure 5.

The intuition behind the shape of $f_2(x)$ as displayed in Figure 5 is the following: When the inventory level is at a moderate range between (approximately) 0.5 and 1.5, it may

Table 2. Average cost function(s) for Model II evaluated at different values of q. For the a values indicated we have $\int_0^q f_2(x)dx = 1$. The optimal values are underlined.

q	a	Average HC	Average SC	Average CC	C_2(q)
1.00	0.595	0.49	0.51	2.38	3.38
2.00	0.274	0.99	0.44	1.09	2.52
2.10	0.258	1.04	0.43	1.04	2.51
2.19	0.246	1.08	0.42	0.99	2.49
<u>2.20</u>	0.245	1.08	0.42	0.98	<u>2.48</u>
2.21	0.244	1.10	0.42	0.97	2.49
2.30	0.234	1.20	0.41	0.89	2.50
2.40	0.224	1.22	0.41	0.88	2.51
3.00	0.176	1.48	0.40	0.70	2.58

stay in that range with high probability. This is so since the "all-or-nothing" type demand faced in this model implies that even though there may be some inventory available, a new demand may not be satisfied if it exceeds the available amount thus keeping the inventory intact.

4. Model III: Sporadic Review with Completely or Partially Satisfied ("All-or-Some") Demand

Consider now the reflected process **W** in (2) but under the random clearing policy $\tau = \tau(\xi) \sim \exp(\xi)$ which is independent of $\{W(t) : 0 \leq t \leq \tau(\xi)\}$. The content level process

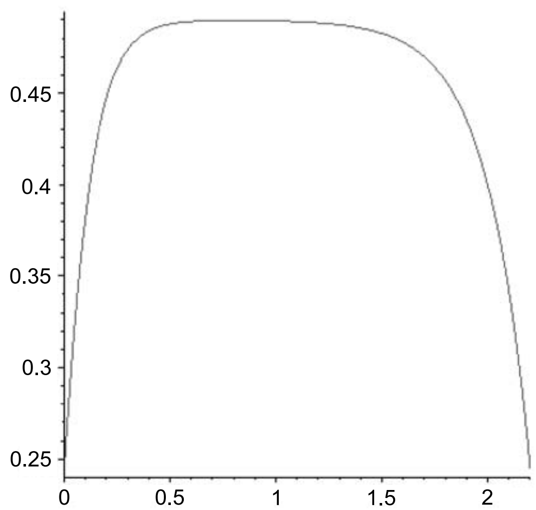

Figure 5. Stationary density $f_2(x)$ of the content level process when $q^* = 2.20$ with $f_2(0) = f_2(q^*) = 0.245$.

in the buffer $\mathbf{V}_3 = \{V_3(t) : t \geq 0\}$ is a regenerative process with cycle $\tau(\xi)$ whose sample path is the stochastic replication of the family $\{W(t) : 0 \leq t \leq \tau(\xi)\}$. In this model variant we extend the assumption of exponential jumps in the buffer. We assume that the jumps are i.i.d. random variables having distribution $G(\cdot)$, density $g(\cdot)$, mean $1/\mu$ and LT $\tilde{g}(\alpha) = \int_0^\infty e^{-\alpha t} g(t)\, dt$. Typical realizations of $\tau(\xi)$ and $\mathbf{V}_3$ are depicted in Figure 6. Interestingly, despite the simplicity of the result in the next Lemma, it is impossible (to the best of the authors' knowledge) to obtain it analytically. The result is obtained by an educated guess.

4.1. Stationary Distribution of the Content Level $V_3(t)$

The next Lemma provides an explicit expression for the distribution of the content level process.

LEMMA 1 *The content level process of* $\mathbf{V}_3$ *in steady state is exponentially distributed with parameter* $\eta(\xi)\left[i, e., f_3(x) \equiv f_{V_3}(x) = \eta(\xi) e^{-\eta(\xi)x}\right]$ *where* $\eta = \eta(\xi)$ *is the unique root of the equation*

$$\eta - \lambda[1 - \tilde{g}(\eta)] + \xi = 0.$$

Proof: The appropriate steady state (Pollaczek–Khintchine-type) equation of Model III is

$$f_3(x) = \lambda \int_x^\infty [1 - G(w - x)] f_3(w)\, dw + \xi \int_x^\infty f_3(w)\, dw. \tag{13}$$

The left hand side of (13) is the long-run average number of up-crossing of level x. Thus, the right hand side must be equal to the long-run average number of up-crossing. The

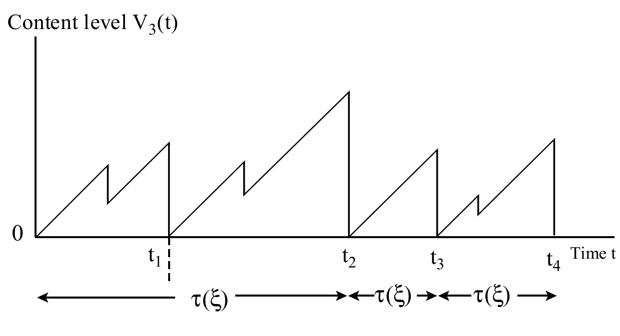

Figure 6. Sample paths for the inventory level in Model III [sporadic review with completely or partially satisfied ("All-or-Some") demand]. At epochs t_2, t_3 and t_4 the buffer is cleared and at epoch t_1 some of the demand at the buffer (indicated by the dotted lines) is lost. The cycle length is $\tau(\xi)$.

parameters λ and ξ are the rates of the Poisson demand arrivals and the Poisson clearing arrivals, respectively. The conditional probability to down-cross level x given $\mathbf{V}_3 \in dw$ by a regular demand is

$$\Pr(S > w - x) = 1 - G(w - x)$$

and at a moment of clearing is $\Pr(S > x)$. Finally, by PASTA, the density $f_3(\cdot)$ appears both in the left hand side and in the right hand side of (13).

We now try the "educated guess"

$$f_3(x) = \eta e^{-\eta x} \tag{14}$$

for some positive η. Substituting (14) in (13), multiplying both sides by $e^{\eta x}$, and then subtracting ξ, we get

$$\eta - \xi = \lambda\eta \int_x^\infty e^{-\eta(w-x)}[1 - G(w - x)]\, dw. \tag{15}$$

But

$$\int_x^\infty e^{-\eta(w-x)}[1 - G(w - x)]\, dw = \frac{1 - \tilde{g}(\eta)}{\eta}.$$

Thus

$$\eta = \lambda[1 - \tilde{g}(\eta)] + \xi. \tag{16}$$

To show that η is unique define the functions $\ell(\eta) = \eta - \xi$ and $h(\eta) = \lambda[1 - \tilde{g}(\eta)]$. It is easy to see that $h(0) = 0$, $h(\infty) = \lambda$ and that $h(\eta)$ is concave increasing in η. Also, $\ell(0) = -\xi$ and $\ell(\infty) = \infty$. Thus, by equating $\ell(\eta) = h(\eta)$ there must be a unique match point $\eta \in (0, \lambda)$. By the limit theorem for regenerative processes the stationary distribution is unique. Thus, according to the educated guess (14), the stationary distribution of $\mathbf{V}_3$ is $\exp(\eta)$ and by PASTA the amount cleared is also $\exp(\eta)$. The proof is complete. ■

It is worth noting that in a recent paper Kella and Miyazawa (2001) also use an "educated guess" to show that the steady state law of the conditional $G/M/1$ queue in which the idle periods are deleted is exponential. However, despite the similarity of the guess used, the model presented here and Kella and Miyazawa's (K–M) model are not exactly the same implying that the parameters of the exponential distributions found by K–M and by us are not the same.

4.2. Optimization of Model III

To develop the average cost function for Model III, we use essentially the same arguments that were used in the development of the average cost function for Model I in

Section 2.2. Noting that the expected cycle length is now $1/\xi$ [rather than $1/f_1(q)$], we obtain the objective function for Model III as

$$C_3(\xi) = h\int_0^\infty x f_3(x)\,dx + \frac{\pi\lambda}{\mu}\int_0^\infty [1 - G(x)] f_3(x)\,dx + K\xi, \tag{17}$$

where the density $f_3(x)$ is a function of the decision variable ξ.

Consider again the data values that were used in previous models, i.e., $(\lambda, \mu;\ h, \pi, K)$ = (5,10;1,2,4). With these values we find the LT as $\tilde{g}(\eta) = 10/(\eta + 10)$, and solving (16) for η gives

$$\eta(\xi) = \frac{1}{2}\left(\xi - 5 + \sqrt{25 + 30\xi + \xi^2}\right) \geq 0,$$

so that $f_3(x) = \eta(\xi)e^{-\eta(\xi)x}$. Substituting $f_3(x)$ into (17) we obtain a convex function

$$C_3(\xi) = \frac{2\left[40 - 89\xi + 81\xi^2 + 4\xi^3 + \sqrt{\Delta(\xi)}(-4 + 21\xi + 4\xi^2)\right]}{\left[15 + \xi + \sqrt{\Delta(\xi)}\right]\left[-5 + \xi + \sqrt{\Delta(\xi)}\right]\xi}$$

where $\Delta(\xi) \equiv 25 + 30\xi + \xi^2$; see Figure 7 for a plot of $C_3(\xi)$. The value that minimizes this average cost function is found as $\xi^* = 0.34$ for which the minimum average cost is $C_3(\xi^*) = 2.97$ and $\eta(\xi^*) = 0.64$.

Note that the exponential stationary density $f_3(x) = 0.64e^{-0.64x}$ implies that in this model there is a very small probability that the inventory level will be very high since

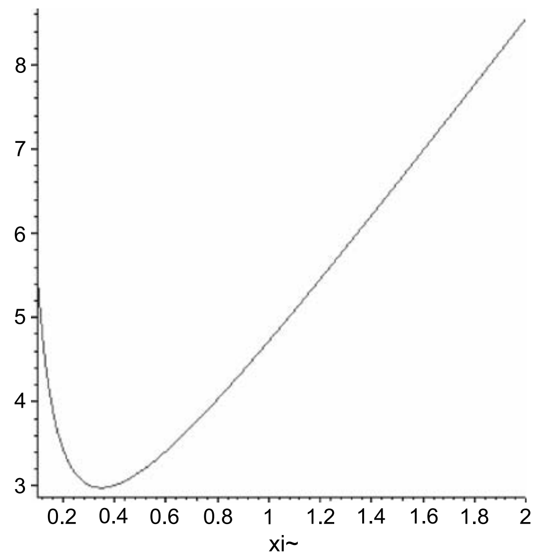

Figure 7. Average cost function $C_3(\xi)$ for Model III when $(\lambda, \mu;\ h, \pi, K)$ = (5,10; 1, 2, 4). This function is minimized at $\xi^* = 0.34$ which gives $C_3(\xi^*) = 2.97$.

either the buffer will be cleared by the controller at random intervals or the inventory will be depleted by "all-or-some" type demands. Also note that since η is increasing in ξ, and the expected cycle length is decreasing in ξ, higher values of ξ result in a lower average inventory since the buffer is cleared more frequently by the controller.

5. Model IV: Sporadic Review with Completely Satisfied or Completely Unsatisfied ("All-or-Nothing") Demand

In Model IV the content level process $\mathbf{V}_4$ in the buffer is no longer a continuous time random walk. The demands are, as in Model II, of the "All-or-Nothing" type in the sense that each demand is either completely satisfied (if its size is smaller than the content level); or completely unsatisfied, otherwise. Clearly, level 0 can be reached by $\mathbf{V}_4$ only at moments of clearings that arrive in accordance with a Poisson process of rate ξ. That is, $\mathbf{V}_4$ is a regenerative process whose cycle $\tau(\xi) \sim \exp(\xi)$ is independent of $\{V_4(t) : 0 \le t < \tau(\xi)\}$. For a sample realization of the content level process see Figure 8.

Let $L_1, L_2, \ldots$ be the $\exp(\lambda)$ interarrival times and $Z_1, Z_2, \ldots$ be the $\exp(\mu)$ jump sizes. We describe the dynamics of $\mathbf{V}_4$ in the first cycle as follows:

$$V_4(t) = \gamma t, \quad \text{for } 0 \le t < \min(L_1, \tau(\xi)),$$

$$V_4(\min(L_1, \tau(\xi))) = \begin{cases} 0, & \text{if } \tau(\xi) \le L_1 \\ \gamma L_1 - Z_1 \cdot \mathbf{1}_{\{Z_1 \le \gamma L_1\}}, & \text{if } \tau(\xi) > L_1. \end{cases}$$

If $\{\tau(\xi) \le L_1\}$, the cycle is terminated. If $\{\tau(\xi) > L_1\}$ the cycle is not yet terminated and

$$V_4(t) = V_4(L_1) + \gamma(t - L_1), \quad \text{for } L_1 \le t < \min(L_1 + L_2, \tau(\xi)),$$

$$V_4(\min(L_1 + L_2, \tau(\xi))) = \begin{cases} 0, & \text{if } \tau(\xi) \le L_1 + L_2 \\ V_4(L_1) + \gamma L_1 - Z_2 \cdot \mathbf{1}_{\{Z_2 \le V_4(L_1) + \gamma L_2\}}, & \text{if } \tau(\xi) > L_1 + L_2. \end{cases}$$

With these preliminaries, we now turn to the analysis of the buffer level distribution.

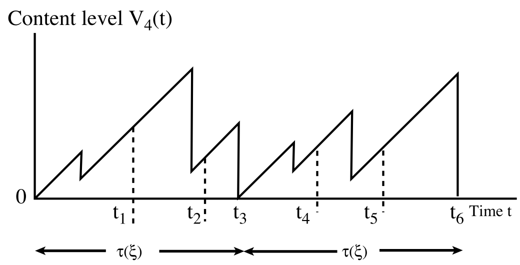

Figure 8. Sample paths for the inventory level in Model IV [sporadic review with completely satisfied or completely unsatisfied ("All-or-Nothing") demand]. At epochs t_3 and t_6 the buffer is cleared and at epochs t_1, t_2, t_4 and t_5 all demand at the buffer (indicated by the dotted lines) is lost. The cycle length is $\tau(\xi)$.

5.1. *Stationary Distribution of the Content Level $V_4(t)$*

The steady state (Khintchine–Pollaczeck) integral equation for the density $f_4(x) \equiv f_{V_4}(x)$ of $\mathbf{V}_4$ is given by

$$f_4(x) = \lambda \int_x^\infty \left[e^{-\mu(w-x)} - e^{-\mu w} \right] f_4(w)\, dw + \xi \int_x^\infty f_4(w)\, dw \tag{18}$$

where the intuitive explanation of (18) is similar in nature to that of the previous section.

The stationary density of the buffer's content level, $f_4(x)$, is found as the solution of the following second order ordinary differential equation

$$f_4''(x) + [\lambda(1 - e^{-\mu x}) + \xi - \mu] f_4'(x) + (2\lambda\mu e^{-\mu x} - \mu\xi) f_4(x) = 0 \tag{19}$$

with the initial/boundary conditions $f_4(0) = \xi$ and $\int_0^\infty f_4(x)dx = 1$.

We now solve the ODE in (19) to find the stationary density $f_4(x)$ which is obtained in terms of some special functions. First, we write (19) as

$$f_4''(x) + (ke^{-\mu x} + \ell) f_4'(x) + (ue^{-\mu x} + v) f_4(x) = 0 \tag{20}$$

where $k \equiv -\lambda$, $\ell \equiv \lambda + \xi - \mu$, $u \equiv 2\lambda\mu$ and $v \equiv -\mu\xi$ are constants (for a given ξ). Solving the ODE in (20) with the help of the computer algebra system Maple (Heal et al., 1998) we obtain

$$\begin{aligned} f_4(x) = {} & c_1 \exp\left[\frac{1}{2\mu}(\mu x(\mu - \ell) + ke^{-\mu x})\right] \mathcal{W}_W\left(\frac{k(\mu-\ell) + 2u}{2\mu k}, \frac{1}{2\mu}\sqrt{\ell^2 - 4v}, \frac{1}{\mu} ke^{-\mu x}\right) \\ & + c_2 \exp\left[\frac{1}{2\mu}(\mu x(\mu - \ell) + ke^{-\mu x})\right] \mathcal{W}_M\left(\frac{k(\mu-\ell) + 2u}{2\mu k}, \frac{1}{2\mu}\sqrt{\ell^2 - 4v}, \frac{1}{\mu} ke^{-\mu x}\right), \end{aligned}$$

where c_1 and c_2 are the arbitrary constants to be determined using the conditions $f_4(0) = \xi$ and $\int_0^\infty f_4(x)\, dx = 1$, and the independent solutions $\mathcal{W}_W(a,b,x)$ and $\mathcal{W}_M(a,b,x)$ are the "Whittaker W" and "Whittaker M" functions, respectively. [To check this solution, we substituted it in (20) and found that it does indeed satisfy the ODE.] These special functions are given as

$$\begin{aligned} \mathcal{W}_W(a,b,x) &= e^{-x/2} x^{1/2+b} \mathcal{K}\left(\frac{1}{2} + b - a, 1 + 2b, x\right), \\ \mathcal{W}_M(a,b,x) &= e^{-x/2} x^{1/2+b} \mathcal{H}\left(\frac{1}{2} + b - a, 1 + 2b, x\right), \end{aligned}$$

with the Kummer function $\mathcal{K}(a,b,x)$ being one of the independent solutions of another 2nd order ODE [i.e., $xy''(x) + (b - x)y'(x) - ay(x) = 0$] and the hypergeometric function

$$\mathcal{H}(a,b,x) = \sum_{n=0}^\infty \frac{\Gamma(a+n)/\Gamma(a)}{\Gamma(b+n)/\Gamma(b)} \left(\frac{x^n}{n!}\right)$$

where $\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt$ is the gamma function evaluated at $a > 0$; (Abramowitz and Stegun, 1965, Ch. 13).

It is important to point out that although these special functions are defined in terms of infinite sums and solution of a differential equation, they have been implemented in many computer algebra systems such as Maple (Heal et al., 1998) which makes their computations relatively straightforward.

5.2. Optimization of Model IV

To develop the average cost function for Model IV, we use arguments similar to those that were used in the development of the average cost function for Model II in Section 3.2. The objective function for Model IV is found as

$$C_4(\xi) = h \int_0^\infty x\, f_4(x)\, dx + \pi\lambda \left[\frac{1}{\mu} + I_4(\xi) \right] \int_0^\infty [1 - G(x)] f_4(x) dx + K\xi$$

where $I_4(\xi) \equiv \int_0^\infty x f_4(x)\, dx$ since the form of $f_4(x)$ depends on the choice of the decision variable ξ.

Consider again the problem with parameter values $(\lambda,\ \mu;\ h, \pi, K) = (5, 10;\ 1, 2, 4)$. Solving the ODE with these parameters and the initial/boundary conditions for a *given* ξ, we can find the density $f_4(x)$ *exactly* in terms of the Whittaker function as discussed above. However, since the density $f_4(x)$ is obtained only after specifying the decision variable ξ, as in Model II, we need to perform a line search to determine the optimal ξ value. The results (to two significant digits) of this analysis are summarized in Table 3 where we observe that the optimal solution is $\xi^* = 0.17$ with the minimum average cost of $C_4(\xi^*) = 6.84$.

For the optimal value of $\xi^* = 0.17$, the density $f_4(x)$ of the buffer is plotted in Figure 9. The form of the stationary density $f_4(x)$ implies that in this model there is a very small probability that the inventory level will be very high or very low. This is so since the buffer will be cleared by the controller at random intervals (thus reducing the possibility of very high inventory). At the same time, there is a low probability that the inventory will be very small since the "all-or-nothing" type demand faced in this model implies

Table 3. Average cost function(s) for Model IV evaluated at different values. The optimal values are underlined.

ξ	Average *HC*	Average *SC*	Average *CC*	$C_4(\xi)$
0.10	5.10	2.13	0.40	7.63
0.15	3.48	2.81	0.60	6.89
0.16	3.27	2.95	0.64	6.86
0.17	3.08	3.08	0.68	6.84
0.18	2.92	3.21	0.72	6.85
0.19	2.78	3.34	0.76	6.88
0.25	2.14	4.12	1.00	7.26

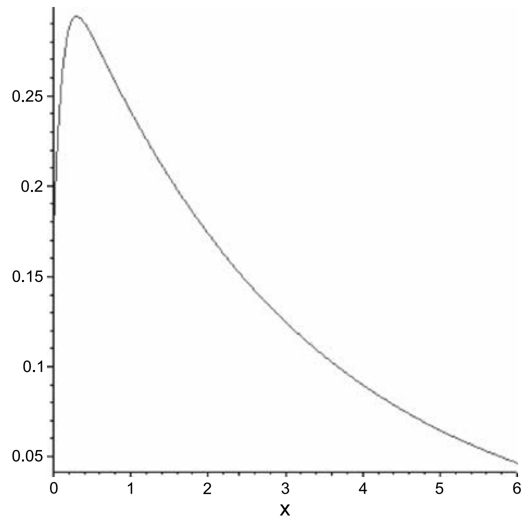

Figure 9. Stationary density $f_4(x)$ of the content level process when $\xi^* = 0.17$.

that even though there may be some inventory available, a new demand may not be satisfied if it exceeds the available amount thus keeping the inventory intact.

6. Summary and Conclusions

We examined four models arising from different combinations of review timing (continuous vs. sporadic) and the amount of demand satisfied ("All-or-Some" vs. "All-or-Nothing"). We first determined the integral equations for the stationary distributions of the content level process for each model using arguments from level crossing theory. The integral equations were solved analytically in the case of Models I and III, and numerically in the case of Models II and IV. After determining the stationary distributions, we formulated optimization models for each case and found the optimal value of the decision variables (q or ξ) to minimize an average cost objective function.

Table 4. Summary of the average holding, shortage and clearing cost expressions. The sum of these costs gives the total average cost.

Model	Average *HC*	Average *SC*	Average *CC*
I	$h\int_0^q xf_1(x)dx$	$\frac{\pi\lambda}{\mu}\int_0^q [1-G(x)]f_1(x)dx$	$Kf_1(q)$
II	$h\int_0^q xf_2(x)dx$	$\pi\lambda\left[\frac{1}{\mu}+I_2(q)\right]\int_0^q [1-G(x)]f_2(x)dx$	$Kf_2(0) = Kf_2(q)$
III	$h\int_0^\infty xf_3(x)dx$	$\frac{\pi\lambda}{\mu}\int_0^\infty [1-G(x)]f_3(x)dx$	$K\xi$
IV	$h\int_0^\infty xf_4(x)dx$	$\pi\lambda\left[\frac{1}{\mu}+I_4(\xi)\right]\int_0^q [1-G(x)]f_4(x)dx$	$K\xi$

Table 5. Summary of the results of the optimization analyses for the four models.

Model	Decision	Minimum average cost
I	$q^* = 2.15$	$C_1(q^*) = 2.05$
II	$q^* = 2.20$	$C_2(q^*) = 2.48$
III	$\xi^* = 0.34$	$C_3(\xi^*) = 2.97$
IV	$\xi^* = 0.17$	$C_4(\xi^*) = 6.84$

We summarize the average holding cost (HC), shortage cost (SC) and clearing cost (CC) terms for each model in Table 4.

Table 5 includes summary information on the numerical examples we presented. Note that as we move from Model I to Model II, the "risk" in the system increases as any shortage results in a total loss of all demand. To compensate for this, the optimal solution in Model II takes a value that is larger than that found in Model I. Similarly, as we move from Model III to Model IV, the "risk" of demand losses again increases. We thus find that in Model IV it is optimal to use, on average, a longer review period ($1/\xi^* = 5.88$) than that found for Model III ($1/\xi^* = 2.94$).

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