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MOTIVATING CONSTRAINTS OF A PEDAGOGY-EMBEDDED COMPUTER ALGEBRA SYSTEM

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ABSTRACT. The constraints of a computer algebra system (CAS) generally induce limitations on its usage. Via the pedagogical features implemented in such a system, "motivating constraints" can appear, encouraging advanced theoretical learning, providing a broader mathematical knowledge and more profound mathematical understanding. We discuss this issue, together with two examples from Calculus, which show an important feature of an instrumentation process.

KEY WORDS: computer algebra system, constraints, instrumentation, learning

Introduction

Computer algebra systems were originally designed in order to perform technical computations to be used, for example, in situations where tables of integrals, Laplace transforms, etc. were formerly used. This usage was intended to replace hand-made computations. Over time, these systems evolved, and educators discovered pedagogical applications and implemented various educative activities based on their usage. In mathematics education, purely technical usage is of minimal interest, emphasis is put on mathematical meaning and conceptual understanding.

The functions of a computer algebra system (a CAS) as an assistant to mathematical learning are placed into three levels:

1. A technical tool performing technical tasks
2. A tool whose performance helps to develop more conceptual understanding
3. A technological aide to bypass a lack of conceptual knowledge, where such knowledge is out of reach, at least in "the immediate future".

The first level is the blackbox level and has no great pedagogical value, beyond saving time and technical efforts. Maybe it allows the teacher to save time for reflexion and theoretical understanding, but a perverse effect is the loss of manual computation skills, as noted by (Herget, Heugl, Kutzler, & Lehmann, 2000). This effect looks like the simple calculator

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(where only the so-called four operations are implemented) introduced in primary schools, whereby mental calculation almost disappeared. For high-school students and undergraduates, other abilities are about to disappear, such as integration techniques, or techniques for solving equations, either linear or non-linear. Only "specialists" (those who program the computers) will master the theory, and not only the know-how.

At level $1\ ^1/_2$, the student uses the CAS for verifying results. There are two kinds of verifications:

- Verify either a numerical result or a "closed" algebraic expression.
- Perform the passage from n to $n + 1$ in a recurrence, after the CAS enabled conjecture of a formula (see Garry, 2003, p. 139).

For the first case, the mathematical correctness of the verification is not always evident. For example, two different CAS or even two different commands of the same CAS, or a CAS and hand-work, can provide different algebraic expressions, both valid. As inert expressions, they are different, but when defining functions, which are dynamic objects, different expressions can define the same function. This can be a good opportunity to recall the definition of two equal functions. The verification issue has been addressed by Lagrange (1999) and Pierce (2001).

Steiner & Dana-Picard (2004) commented on aspects of level 2. Low-level commands are important for cognitive processes attempting to afford a good conceptual insight. A CAS command is called a *low-level* command if it performs a single operation, while a macro is a command programmed to perform a sequence of low-level commands. Perhaps we should consider low-level commands as the atoms of every computerized process aimed to solve a mathematical problem.

Because of the limitations imposed by the syllabus and also because of time limitations, level 3 is less commonly considered. However, it can be included close to the frontier of the syllabus, either for enriching exercises, or for problem solving when the necessary theorems have still not been taught (see Dana-Picard, 2005a).

In this paper we wish to show that a fourth level exists: a CAS is a device whose performances sometimes *force* the user to acquire more mathematical knowledge. Perhaps there exists more than one command, more than one algorithm, to solve a problem and a total freedom of choice is left to the user. For the user to make an intelligent decision, he/she must have a good knowledge of the mathematics implemented in these algorithms. Of course, there exist situations where a unique algorithm is available, either because of the theoretical state-of-the-art or because of the decisions of the developers. This limits the diversity offered by the CAS; this issue is studied by Artigue (2002, p. 265).

In every case, the implemented mathematics has to be understood if one desires to master what is at work there. In the next sections, we show examples where the CAS commands are based on a theorem which does not appear in a standard syllabus. In order to afford a real understanding of the process, the user has to learn new mathematics. We will call such a situation a *motivating constraint* of the software. It pushes the user towards a more profound mathematical insight.

Until recently, the choice of which feature of the CAS, which command, to use was completely left to the user and no guidance was provided by the CAS itself. This claim is not true anymore: pedagogical features have been implemented into computer algebra systems. We will call such systems *pedagogy embedded CAS*. For example, Derive 6 has a *step-by-step* feature, run by a simple left-button click. Every step corresponds to what we call low-level commands, either because it could have been programmed as a single independent command, or because it implements one theorem (an integration formula, an elementary operation on rows of a matrix, etc.). We will show surprising situations. Let us only mention that this *step-by-step* feature is still under development. Other systems are also pedagogy embedded. For example Maple is pedagogy embedded (with the *Student* package and other features, such as the tutorials of version 10). Its functionalities work in a different way than those of Derive, and the learning process induced by them develops otherwise. We do not wish to compare here the two embeddings of pedagogical features; the comparison deserves a special study.

As noted by Trouche (2004b), tools *shape* the learning environment. The actual issue we wish to address here is one influence of the computerized environment on the mathematical contents. We discuss two examples from integral calculus at the undergraduate level. Please note that the software follows general algorithms, starting from pattern recognition, and whose sequential steps are based on the implementation of general theorems. The human brain works less sequentially, therefore intuition can lead to other pathways towards the solution of the exercise. Comparing the two ways is very productive.

Should we temperate slightly the previous claim? Generally, the human brain does not work sequentially, but in certain situations it is wise to translate the solution process of a given problem into a flowchart, then into an algorithmic process, as shown by Meyer & Dana-Picard (1997).

This occurs when working with students having special difficulties with problem solving. We do not deal with such a situation here.

TWO DIFFERENT ACTIVITIES AROUND THE SAME EXAMPLE OF A MATHEMATICAL TOPIC

The Mathematical Situation

We consider the definite integral $I_n = \int_0^{\pi/2} \sin^n x\,dx$, where the parameter n is a non negative integer. Asking the computer to compute I_n for general parameter provides an output identical to the input, indicating that the software failed to compute the given parametric integral.

Working with paper-and-pencil, we find a recurrence relation with I_n as a function of n and I_{n-2}, namely, $I_n = \frac{n-1}{n} I_{n-2}$. Detailed computations are given in Dana-Picard (2004). The fact that the index increases by 2 shows that the sequence of integrals splits into two distinct subsequences: the terms with odd index, and the terms with even index.

Case Description: A Question in a Calculus Exam

Computation of $I_3 = \int_0^{\pi/2} \sin^3 x\,dx$ was a question as part of an exercise in a recent Calculus examination at Jerusalem College of Technology involving 116 students. The students had to solve five exercises among seven proposed. With respect to this question, the students dispatched as follows:

- 54 students answered the exercise containing this question
- 10 of them gave a wrong answer, generally because of mistakes in technicalities
- The solutions were dispatched into three very different groups:

(i) A little less than two thirds of the students chose the following way:

$$\begin{aligned} I_3 &= \int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \sin^2 x \cdot \sin x\,dx = \int_0^{\pi/2} (1-\cos^2 x)\cdot \sin x\,dx \\ &= \int_0^{\pi/2} \sin x\,dx - \int_0^{\pi/2} \cos^2 x \cdot \sin x\,dx \\ &= [-\cos x]_0^{\pi/2} + \left[\tfrac{1}{3}\cos^3 x\right]_0^{\pi/2} = 1 - \tfrac{1}{3} = \tfrac{2}{3}. \end{aligned}$$

In this group, half of the students computed the integral as shown here, the other half made an explicit substitution for the integral on the right of the minus sign.

(ii) About one third of the students began with the same decomposition

$$I_3 = \int_0^{\pi/2} \sin^3 x\, dx = \int_0^{\pi/2} \sin^2 x \cdot \sin x\, dx,$$

then performed an integration by parts. This is allowed, but the computation process is more complicated than in the previous way. Together with other examples, students showed a preference for integration by parts, rather than substitution, when dealing with a definite integral.

(iii) Four students tried a substitution. Two of them had problems with the new boundaries of the integral, the other two computed an indefinite integral using a substitution, then used the result to compute the requested definite integral.

(iv) One student proved the general recurrence formula for I_n, then computed the first terms of the sequence and finally obtained the desired value for I_3. His motivation for this choice is easy to understand: a similar exercise, deriving a general recurrence formula, had been solved in classroom a short time before the exam. We wish to point out that this student probably has a good memory, but poor mathematical skills.

Educative Activities

Let us now describe two different educative activities with Derive, based on the computation of this parametric integral. The first one does not include Derive's *step-by-step*, but the second one uses it.

(a) Without *step-by-step*: We use the standard commands with immediate answer.

(i) Compute the integral I_n for a possibly wide range of values of the parameter. The VECTOR command fits.

$$\#4 \quad \text{VECTOR}\left(\int_0^{\pi/2} \text{SIN}(x)^n dx,\ n,\ 1,\ 10\right)$$

$$\#5 \quad \left[1, \frac{\pi}{4}, \frac{2}{3}, \frac{3\cdot\pi}{16}, \frac{8}{15}, \frac{5\cdot\pi}{32}, \frac{16}{35}, \frac{35\cdot\pi}{256}, \frac{128}{315}, \frac{63\cdot\pi}{512}\right]$$

(ii) Try to find a general formula corresponding to the values that have been obtained (In our current example, the first remark is that they are actually two subsequences, one of rational numbers, the other of multiples of π by rational numbers, according to the parameter being either even or odd; therefore, two formulas have to be conjectured).

(iii) Extra help may be needed for a formula to be conjectured. Here the Online Encyclopedia of Integer Sequences provides this help.

(iv) After a formula has been conjectured, it should be checked for other values of the parameter, possibly using the VECTOR command once again, with another range for the parameter values.

(b) With *step-by-step*: Steiner & Dana-Picard (2004) show one case where Derive can compute a parametric integral for general parameter. Nevertheless, the software cannot always compute such an integral for general parameter. Therefore we propose the following canvas.

(i) Compute the integral I_n for a few small values of the parameter, say $n = 1,2,3,4$, as in Appendix 1.

(ii) Note that in each of the corresponding sessions, the same theorem is applied.

(iii) Use this central formula in order to find the recurrence relation quoted above.

(iv) Find a closed form for I_n using factorials.

(v) Check the correctness of this form for a large range of values of the parameter using, for example, the VECTOR command of Derive.

Discussion

The first activity provides a proof of a combinatorial expression for the given integral. It is a nice opportunity for the educator to have students search a question, propose a conjecture, and try to verify it. Creative activities of this type are a plus in the face of the imitative processes most frequently met in regular curricula.

The second activity also provides a proof of the formula which has been discovered, but it has an added value: the discovery of a non-standard theorem. A Derive 6 session for computing I_3 is displayed in Appendix 1. Pay attention to the following row:

$$\int \mathrm{SIN}(a\cdot x+b)^p dx \rightarrow -\frac{\mathrm{SIN}(a\cdot x+b)^{p-1}\cdot \mathrm{COS}(a\cdot x+b)}{a\cdot p} + \frac{p-1}{p}\cdot \int \mathrm{SIN}(a\cdot x+b)^{p-2} dx$$

This formula is obtained by integration by parts, exactly in the way the recurrence relation of the first activity has been obtained. Generally this theorem does not belong to the Calculus syllabus. At most, computation of parametric integrals is given as an advanced exercise for advanced students, definitely not within the course's mainstream. If the theorem did not appear explicitly in the computerized process, it would probably not have been shown, a fortiori not be proven in classroom. We discuss this point in the last section. Note that the core influence of the CAS lies in points (i) and (ii).

We should mention that, during the examination, only one student chose the way the software computes the integral for pencil-paper work.

Second Example: Computation of a Parametric Integral for General Parameter Value

The example here is more striking. We consider the improper integral $I_r = \int_0^{\pi/2} \frac{1}{1+\tan^r x} dx$, where r is a non negative real parameter. This integral is equal to $\pi/4$ for any non negative real value of the parameter. It can be computed by hand, in a small number of steps (see Steiner and Dana-Picard, 2004). Steiner proposed this in a Calculus exam, with r equal to the year's number.

In most situations, a CAS does not work with general parameter. The student/mathematician has to make computations for various values of the parameter, then conjecture a formula, and finally prove the conjecture, as already mentioned in the previous section.

When computing this specific integral, Derive works otherwise: it computes the improper integral as a definite one, using a pathway totally different from the hand-work pathway mentioned above. The Derive session is displayed in Appendix 2. During the exam, no student solved the question in the way the computer uses here.

Herget et al. (2000) list the important skills and abilities that a student must have when trying to solve an exercise, either manually or using a CAS. We mention here what is relevant to our goal:

- Finding expressions
- Recognizing structures
- Visualizing
- Properly using the technology

Both for manual computation and computerized work, the core item is the recognition of the rth power of the tangent function. This power is transformed by Derive into an exponential and the rest follows. This recognition, very different from the human recognition, explains why the computation is feasible by the computer for *any* real value of the parameter, without need of an explicit substitution. It seems that the same theorem has not been implemented into other packages. This does not mean that technology has not been programmed properly: *a proper usage of technology does not require the technology to mimic human actions*. Conversely, the human mind does not have to work algorithmically. Therefore, comparison between the two ways is an enriching task.

The computation step which drew our attention is explained by the computer with the following formula (the complete Derive session is given in Appendix 1):

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b (f(x) + f(a+b-x))\, dx$$

This formula is not trivial; comments and examples are given in Dana-Picard (2005b). When looking at these examples, an experienced teacher, not working in the author's College but at a distant University, told the author: "I would not dare to ask my students to know such a theorem." Therefore we wished to check the reactions of a student, named Eytan, what he thought about the situation, and whether to teach or not to teach this integration formula.

At the start, the integral $I_r = \int_0^{\pi/2} \frac{1}{1+\tan^r x}\, dx$ has been proposed to Eytan with general parameter. After a few seconds of reflection, he says: "Let's try with a small parameter value." Eytan writes: $\int_0^{\pi/2} \frac{1}{1+\tan x} dx = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$. Then he says: "I would try a substitution."

Interviewer: Which substitution?
Eytan: $\sin x$. But it doesn't work.

Interviewer: Computing I_0 would help?

Eytan makes the computation. "I get $\pi/4$."
Eytan: I try I_1 in order to try afterwards an induction. I try positive integers, afterwards we'll see other numbers... First let's see what happens, we decide later...

Eytan tries to perform an integration by parts, unsuccessfully.
Interviewer: Maybe you should try I_2?

Eytan: It looks easier.... (he computes)... I get $\pi/4$. Wow, interesting! (wondering a while) It's the same result. I can try I_3. No, it's too hard, I'll try I_4.

Eytan writes $I_4 = \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx$ and stops.

Interviewer: Should I show you a solution?
Eytan: I'll read only one row, and then I'll try alone. Actually Eytan looks at the substitution $u = \pi/2 - x$ and proceeds to computation for general parameter. He obtains: $I_r = \int_0^{\pi/2} \frac{1}{1 + \cot^r x} dx$ and stops.
Interviewer: Do you know any connection between $\tan x$ and $\cot x$?
Eytan: Yes! $1/\tan x$ and $\pi/2 - x$.
Interviewer: Which one do you intend to use?
Eytan: Yes! $\pi/2 - x$.
And he tries$1/\tan x$.

Finally, Eytan reads the entire solution. "Interesting...I would not have thought like that alone..."
Then the interviewer shows the solution with Derive (the step-by-step session). Eytan points out the formula

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b (f(x) + f(a+b-x))\, dx$$

and says: "That's something that he (the computer) has stuck there!"

He thinks a while, draws a graph (Figure 1) in order to visualize what happens (he makes some gestures with his hands, leftwards and rightwards, in order to "feel" the area beneath the graph in both directions, and to visualize what it *means* to substitute $a + b - x$ instead of x) and then writes his own proof of the formula, using a substitution (Figure 2). And he says: "this formula is very interesting. And useful.... I'll use it!"

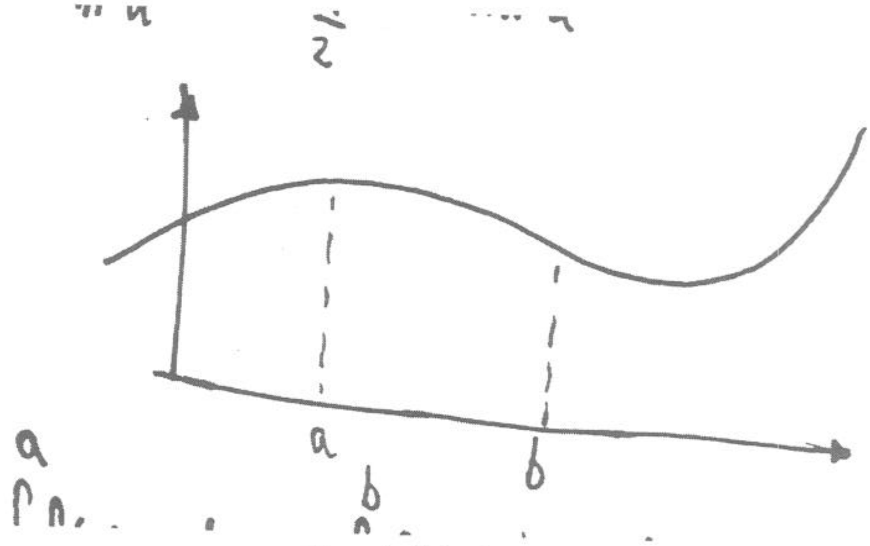

Figure 1. Eythan's graph.

Interviewer: Suppose that you read Mathematics alone and that you meet such a new formula, what do you do?
Eytan: When I learn, I try to understand processes. I would have read what is written, and then I would have tried to make the work by myself.

A couple of days later, the interviewer proposed to Eytan the following integral: $I = \int_0^4 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}}\,dx$. Eytan said immediately "it's the same situation as last time" and computed the integral quickly using the above formula, with the exclamation "really interesting, indeed!".

At the beginning, Eytan's approach is classic, trying the computation for small values of the parameter. His intuition tells him that substitution

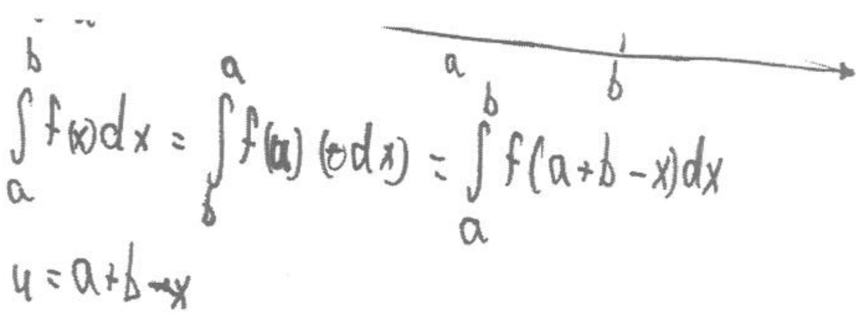

Figure 2. The substitution made by Eythan.

should be a good start, but does not succeed. For him, the presentation of a written solution is not the end of the session, he wants to work by himself. Actually, that's exactly the behavior that an educator would expect from a student, to be active, not to receive a solution in a passive way only.

As a side remark, note that Eytan starts with small *integer* values of the parameter. He intends to check with *other numbers* afterwards. This is a classical attitude with respect to parameters, as most parametric exercises proposed to undergraduates involve non negative integers for parametric values.

The central point in this working session seems to be Eytan's positive reaction when discovering the usefulness of the proposed formula. At first he is somewhat skeptical, but briefly afterwards expresses enthusiasm: "wow, interesting!". The decision to use it beyond this working session is the most important one. Were the formula not shown by the computer, he would not have discovered it in a regular textbook.

General Discussion

Instrumentation

Balacheff (1994) defines the *computerized transposition (transposition informatique)*:

> Je parlerai de transposition informatique pour désigner ce travail sur la connaissance qui en permet une représentation symbolique et la mise en oeuvre de cette représentation par un dispositif informatique, qu'il s'agisse ensuite de la montrer ou de la manipuler (I'll talk about computerized transposition to call this work on knowledge, which provides a symbolic representation and an implementation of this representation by a computerized device, no matter whether the aim is to show it or to manipulate it).

The situation in the second example is one where the knowledge is shown, and then manipulated. In Eytan's working session, manipulation comes partly when he proves the new formula, and more completely with the second integral.

At the beginning, we expected Derive's *step-by-step* to provide the student with "a posteriori assistance", in order to understand what (s)he would have been required to do. This behavior looks like backwards differentiation when checking correctness of an indefinite integration, with a noticeable difference:

- Backwards differentiation checks a result which has been completely computed.

- Here the software provides "post-mortem help" in understanding what should have been done.

Actually, the usage of the *step-by-step* feature of the software can be considered as an "a priori" usage, in one of the following fashions:

- The software user can discover a way of solving the problem different from his/her way.
- If the student did not determine how to solve the problem, the CAS opens a pathway. This is based on general theorems that the student does not automatically know.

Suppose that such a situation occurs in the classroom: the teacher builds various kinds of activities, which enrich by a large amount the mathematical knowledge and culture of the learners. If at the beginning the student influenced the software's behavior in order to obtain the needed result, in the second scenario the software forces the educator to teach and the student to learn a new topic, a new theorem. As A. Rich says:

> The transformation rules Derive displays are those *it* uses to simplify an expression. They may or may not be the same as those currently taught to students. However, if teachers see an advantage to an unfamiliar rule used by Derive, they may want to ask their students to verify the validity of the rule and then the students will have an additional tool in their arsenal (Böhm et al., 2005, p. 36).

We have here elements of an *instrumentation process*, which is part of an *instrumental genesis* (Lagrange, 2000; Artigue, 2002, p. 250; Trouche, 2004a):

> Les potentialités et les affordances d'un artefact (en occurrence le CAS) favorisent le développement de nouveaux schèmes (ou font évoluer les schèmes antérieurs) de résolution d'un type de tâches (ici le calcul d'une intégrale définie) (Trouche, 2005; private e-mail).

Briefly:

> Instrumentation is precisely this part of the process where the artifact prints its mark on the subject (Trouche, 2004b, p. 290).

Of course, this genesis is not reduced to the acquisition and internalization of one single theorem. The present examples are only one occurrence of the mechanisms involved in the process.

The Constraints of the CAS : Limitations and Motivating Effects

Following Balacheff (1994), Guin & Trouche (1999) distinguish three types of constraints of the artifact called *internal constraints* (linked to hardware), *command constraints* (linked to the existence and syntax of the commands), and *organization constraints* (linked to the interface artifact-user). We deal here with a specific appearance of a command constraint, in a very special fashion.

Generally, the word *constraint* evokes a limitation, an impossibility to go beyond a certain borderline. For a software package, this can be a limitation on the size of numbers, on the number of successive parentheses, etc. Among the most documented internal constraints are the finiteness of the screen for graphical applications, and the fact that the real numbers are always approximated by rational numbers. The constraint that we meet here is of a totally different nature: instead of limiting the user within the borders of a certain topic, the CAS demands the user go further, to learn a new theorem or a new technique. It is *a motivating constraint*, which leads to a broadening of the student's mathematical landscape. After its apparition, the mathematical knowledge is not supposed to be shown anymore, the student is incited to learn the new theorem, and then become able to manipulate this knowledge, either with or without the help of the technology.

Another occurrence of a motivating constraint sometimes appears: it occurs that the usage and the non-usage of the *step-by-step* feature yield different outputs, both valid. The student has to perform the transformation of one output into the other. This also broadens mathematical landscape and deepens conceptual understanding. As mentioned by Artigue (2002), this happens already when using a TI-92: the input can be transformed before any command is used, for example, when entering an expression involving the tangent function. The Windows version of Derive does not perform such an automatic transformation, enabling better tracing of the mathematical argument. For example, when looking for an oblique asymptote to the graph of a given function f in a neighborhood of infinity, we need to compute the limit at infinity of $f(x)/x$. These computation steps can be totally traced with the CAS.

Contribution to the Institution's Culture

Following Artigue (2002), we use the word "institution" in a very broad sense. Every educative community with precise rules determining the educative way is an institution: a class is an institution, a course involving several classes is an institution, a College is an institution. Each institution has to *decide* whether to introduce the usage of a CAS in Mathematics courses or not to do so. Not to deal with this issue is also a kind of decision. For example, the institution named JCT decided to teach MatLab and to use it in every engineering course. In a small subset of classes, which can also be viewed as an institution, the author and one of his colleagues, I. Kidron, adopted other packages. For example, a course in Ordinary Differential Equations has been given last year for the first time together with practice sessions based on the usage of Mathematica and Maple.

After a decision to use a CAS has been made, which CAS has to be chosen? Maybe more than one CAS, in order to reap the benefit of more features? In this second case, two different systems will not be used in the same way, students should be acquainted with the usage of one of them, only afterwards with the other one. Which CAS is the first one and which is the second one? This choice is not neutral, it is a fundamental issue. Not only does it fix ways of thinking (the command syntax, possibility of simultaneous representations or not, etc.), but the choice of a CAS imposes the teaching/learning of notions, the introduction of more advanced theorems into the syllabus, as our examples show.

The Jerusalem College of Technology is a College for Engineering. In certain courses, the weight has been placed on applications of Mathematics and on the pragmatic side of technology. The introduction of a pedagogy-embedded CAS has already changed the "institution culture" in certain classes (e.g., the course in Ordinary Differential Equations that we mentioned previously), and is susceptible to change the institution's culture on a larger scale (e.g., all the first year Foundation Courses in Mathematics at JCT):

> Tools are not passive, they are active elements of the culture into which they are inserted. (Noss & Hoyles, 1996, p. 58).

Moreover:

> As regards the objects of knowledge it takes in charge, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meanings of knowing or understanding such or such object. To analyze the life of a mathematical object in an institution, to understand the meaning in an institution of "knowing/understanding" this object, one thus needs to identify and analyze the practices which bring it into play. (Artigue, 2002, p. 248)

Such differences of institutional cultures are evident when considering institutions with different goals, e.g., Teacher Training and Engineer Training. They can appear between two institutions of the same kind, if they chose two different CAS, or did not choose any.

Conclusions

Artigue (2002) notes, after (Bosch & Chevallard, 1999), that the evolution of technology has changed the equilibrium between conceptual and technical work. The embedding of pedagogical references into a computer algebra system is a further step in this change, in favor of conceptual understanding. If, at the beginning, technology was designed to replace the human being for the purely technical part of the work, now it is an integral part of the conceptual reflection and theoretical knowledge.

> Techniques that are instrumented by computer technology are changed: new needs emerge, linked to the computer implementation of mathematical knowledge ... (Artigue, 2002).

We have encountered two such needs, within the same theory. The implementation of powerful theorems into the CAS does not free the user from conceptual work. A contrario, the whole power of this implementation will be revealed by the new task: understanding a new part of the theory.

> ... students encountering a rule with which they are not familiar provide the teacher with a perfect opportunity to ask the students to verify the rule. It seems to me that the ability of students to derive general purpose rules is preferable to their re-deriving special cases of those rules each time a new problem is encountered. Also the recognition that there *are* general purpose rules may be enlightening to some. (A. Rich in Böhm, Rich, Dana-Picard, 2005, p. 37).

The examples studied here show that when a human mind and a CAS solve the same problem, the solution processes can be very different: the human mind approach is guided by his previous knowledge and his own cognitive structures, its own capillary connections (this issue will be addressed in a subsequent paper by the author), the CAS has its built-in features and algorithms. We refer to the diagram in Guin & Trouche (1999, p. 202). A reasonable analysis of the differences allows two openings:

1. The acquisition of new concepts and of new techniques for solving problems
2. A new reflection on the objects already known, and on the novelties discovered during the CAS assisted process

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APPENDIX 1: DERIVE SESSION FOR THE FIRST EXAMPLE

$$\#2: \quad \int_0^{\pi/2} \mathrm{SIN}(x)^3 dx$$

$$\int_a^b F(x)dx \rightarrow \mathrm{SUBST\_DIFF}\left(\int F(x)\, dx,\ x,\ a,\ b\right)$$

$$\#3 \quad \mathrm{SUBST\_DIFF}\left(\int \mathrm{SIN}(x)^3 dx,\ x,\ 0,\ \frac{\pi}{2}\right)$$

$$\int \mathrm{SIN}(a\cdot x+b)^p dx \rightarrow -\frac{\mathrm{SIN}(a\cdot x+b)^{p-1}\cdot \mathrm{COS}(a\cdot x+b)}{a\cdot p} + \frac{p-1}{p}\cdot \int \mathrm{SIN}(a\cdot x+b)^{p-2}dx$$

$$\#4 \quad \mathrm{SUBST\_DIFF}\left(-\frac{\mathrm{SIN}(x)^2\cdot \mathrm{COS}(x)}{3} + \frac{2\cdot\int \mathrm{SIN}(x)dx}{3},\ x,\ 0,\ \frac{\pi}{2}\right)$$

$$\int \mathrm{SIN}(a\cdot x+b)\, dx \rightarrow -\frac{\mathrm{COS}(a\cdot x+b)}{a}$$

$$\#5 \quad \mathrm{SUBST\_DIFF}\left(-\frac{\mathrm{SIN}(x)^2\cdot \mathrm{COS}(x)}{3} - \frac{2\cdot \mathrm{COS}(x)}{3},\ x,\ 0,\ \frac{\pi}{2}\right)$$

$$\mathrm{SUBST\_DIFF}(F(x),\ x,\ a,\ b) \rightarrow F(b) - F(a)$$

$$\#6 \quad \frac{2}{3}$$

APPENDIX 2: DERIVE SESSION FOR THE SECOND EXAMPLE

#1 $\dfrac{1}{1+\mathrm{TAN}(x)^r}$

#2 $\displaystyle\int_0^{\pi/2} \frac{1}{1+\mathrm{TAN}(x)^r}\,dx$

$$\mathrm{TAN}(z) \to \frac{\mathrm{SIN}(z)}{\mathrm{COS}(z)}$$

#3 $\displaystyle\int_0^{\pi/2} \frac{1}{1+\left(\frac{\mathrm{SIN}(x)}{\mathrm{COS}(x)}\right)^r}\,dx$

If $x > 0$,

$$LN(x \cdot z) \to LN(x) + LN(z)$$

#4 $\displaystyle\int_0^{\pi/2} \frac{1}{1+e^{r\cdot(LN(1/\mathrm{COS}(x))+LN(\mathrm{SIN}(x)))}}\,dx$

If $x >= 0$,

$$LN\left(\frac{1}{x}\right) \to -LN(x)$$

#5 $\displaystyle\int_0^{\pi/2} \frac{1}{1+e^{r\cdot(-LN(\mathrm{COS}(x))+LN(\mathrm{SIN}(x)))}}\,dx$

$$\int_a^b F(x)\,dx \to \frac{\int_a^b (F(x)+F(a+b-x))dx}{2}$$

#6 $\displaystyle\int_0^{\pi/2} \frac{1}{2}\,dx$

$$\int_a^b F(x)\,dx \to \mathrm{SUBST\_DIFF}\left(\int F(x)\,dx,\ x,\ a,\ b\right)$$

#7 $\mathrm{SUBST\_DIFF}\left(\displaystyle\int \frac{1}{2}\,dx,\ x,\ 0,\ \frac{\pi}{2}\right)$

$$\int a\,dx \rightarrow a \cdot x$$

$$\#8 \quad \text{SUBST\_DIFF}\left(\frac{x}{2},\ x,\ 0,\ \frac{\pi}{2}\right)$$

$$\text{SUBST\_DIFF}(F(x),\ x,\ a,\ b) \rightarrow F(b) - F(a)$$

$$\#9: \quad \frac{\pi}{4}$$

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