A Theoretical-Computational Conflict

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The first Jerusalem Conference on RESEARCH in MATH EDUCATION
ט"ו אדר תשע"ג
18-19 בפברואר 2013
The conflict

- Paths of light
- Polynomial equations
- Existence of solutions
- The CAS does not find the solutions
The conflict

The reflection law
The conflict
Closed paths of light

• Denote by $\vec{N}_k$ a vector normal to $C$ at the point $P_k$.

• By the reflection law, we have:

$$\angle P_k P_{k-1}, \vec{N}_k = \angle N_k, P_k P_{k+1}$$
The conflict

Triangles of light trapped in a closed Fermat curve

\[ \cos(\overrightarrow{V}, \overrightarrow{PB}) = \frac{-x_0 + (1 - y_0) \cdot \left( \frac{y_0}{x_0} \right)^{n-1}}{\sqrt{1 + \left( \frac{y_0}{x_0} \right)^{2n-2} \cdot \sqrt{x_0^2 + (1 - y_0)^2}}} \]

\[ \cos(\overrightarrow{V}, \overrightarrow{PQ}) = \frac{-2x_0}{\sqrt{1 + \left( \frac{y_0}{x_0} \right)^{2n-2} \cdot 2|x_0|}} \]
The conflict

Polynomial equations

• General equations:

\[
\begin{align*}
x^n + y^n &= 1 \\
(1 - y)(y^{2n-2} - x^{2n-2}) - 2y^{n-1}x^n &= 0
\end{align*}
\]

• The case $n=4$

A Gröbner basis

\[
g_1(x, y) = x^4 + y^4 - 1 \\
g_2(x, y) = 7x^2y - 7x^2 + 4y^{12} - 6y^{11} + 2y^{10} - 4y^9 \\
\quad - 10y^8 + 23y^7 + y^6 + 6y^4 - 21y^3 + 7y^2 - 2 \\
g_3(x, y) = 2y^{13} + 2y^{12} + 4y^{11} + 4y^{10} \\
\quad - 3y^9 - y^8 - 4y^7 - 4y^6 + 3y^5 - 3y^4 - y + 1
\]
The conflict

Where are the solutions?

• Intermediate Value Theorem implies that solutions exist

• The CAS *does not see* the solutions
The conflict

Why does the CAS not see the solutions?

- Fermat’s Last Theorem ensures that the solutions must be irrational
- The algorithm works over the rational numbers we are looking for irrational numbers using a rational device
Different roles for different ways of working

- Paper-and-pencil work: derive a mathematical proof of the *existence* of a solution.

- CAS-assisted part of the work: find an actual *construction* of a solution.
Consequence of the conflict between the theoretical result and the computer output

The student had to perform the following steps:

• Compare the results.

• Try to have a more profound insight into both processes, the paper-and-pencil one (theory) and the CAS assisted one.

• Solve the apparent contradiction.

• Dispatch the final answer in as complete a way as possible, either analytic or graphical.
Danger

A theoretical-computational conflict may lead the student to multiply technical tasks with the CAS (cf Trouche, 2004).
A conflict where one component is graphical

- A logistic differential equation (= a model for the evolution of a population)

\[
\frac{dx}{dt} = rx \frac{K - x}{K}
\]

where \( r \) is the intrinsic growth rate of the population and \( K \) is the "carrying capacity", i.e. the resources can carry a maximal population equal to \( K \).

Example: \( y'(t) = y(t)(1 - y(t)) \)
What is specific in our main example here?

It is a theoretical-computational conflict where the computational part is not graphical but algebraic.
What is the origin of the conflict?

• Tall-Vinner's terminology (1981):
  The student’s concept-image of an irrational number is not so clear at the beginning of his work.

• Zazkis's terminology (2005):
  Students have often only an opaque representation for irrationals.

(What about teachers? 😊)
Constraints

• *internal constraints* (linked to hardware)
• *command constraints* (linked to the existence and syntax of the commands)
• *organization constraints* (linked to the interface artifact-user)
• Here we have a *motivating constraint* (cf Th. D-P. 2007)
Motivating constraint

The conflict may lead to study more profoundly:
- *Rational vs irrational numbers*
- *Fermat Last Theorem*
- *Buchberger algorithm*
- *Etc.*
Another conflict: isoptic curves of an ellipse

• Orthoptic curve of an ellipse = director circle

• Viewing an ellipse under another angle
Bisoptics of an ellipse

• The question: find the geometric locus of points from which a given ellipse is viewed under a given angle.

• The activity around the solution process:
  ➢ Algebraic computations, either by hand or using a CAS
  ➢ Graphical representation of the obtained quartic equation which describes the solution
  ➢ Look for an interpretation using an internet database
  ➢ Solve the conflict!
Examples of isoptic curves of an ellipse
An ellipse with two isoptic curves

$x^4 + 2x^2y^2 + y^4 - \frac{19}{8}x^2 - \frac{49}{8}y^2 + \frac{353}{256} = 0$

Conflict: the database does not fit the result of the computations and the plot

Two kinds of tori

(a) Non self intersecting
   \( R=5, \ r=2 \)

(b) self-intersecting
   \( R=4, \ r=5 \)

(a) The torus

(b) View from inside
Spiric curves = toric intersections
A more theoretical insight into the learning process


Thanks to all the cognitive conflict makers all over the world